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AN INVESTIGATION OF TWO MODELS FOR THE DEGREE VARIANCES OF GLOB--ETC(U)
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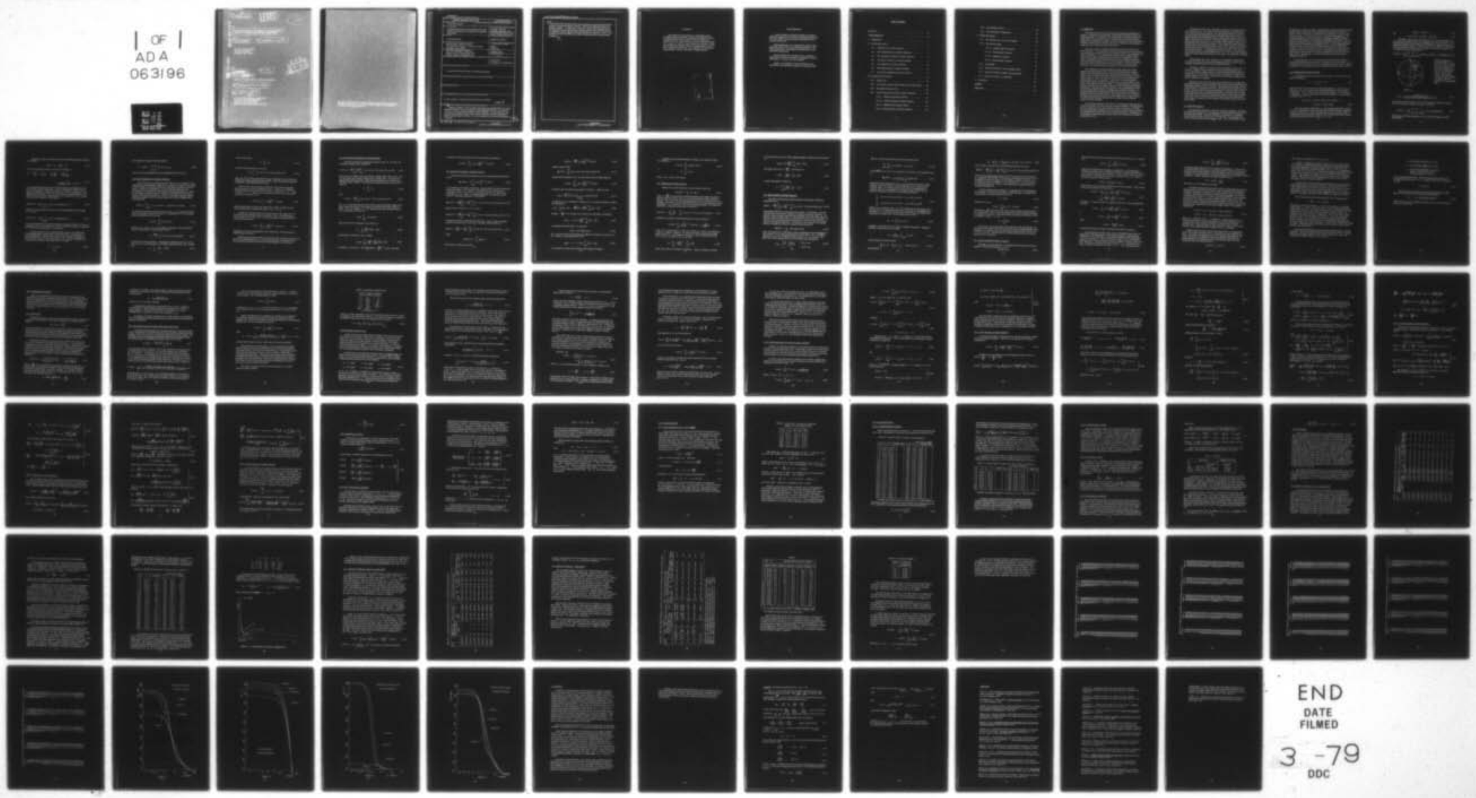

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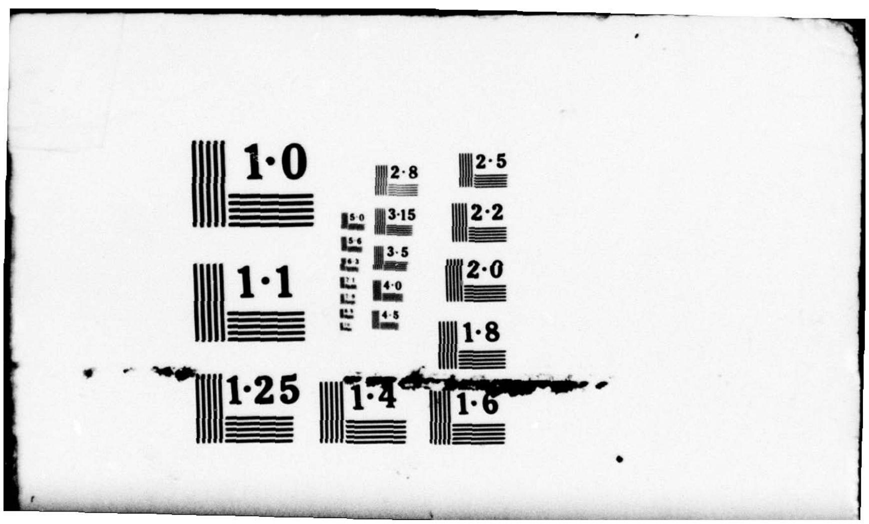
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(18) AFGL (19) TR-78-0235

LEVEL II

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AD A063196

(6) AN INVESTIGATION OF TWO MODELS FOR THE DEGREE VARIANCES OF GLOBAL COVARIANCE FUNCTIONS.

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(9) Interim rept.,

The Ohio State University
Research Foundation
Columbus, Ohio 43212

(11) Sep[redacted]78

(12) 79p.

(14) DGS-275,
Scientific [redacted] 18

Approved for public release; distribution unlimited

(15) F19628-76-C-0010

(16) 7600

(17) 03

AIR FORCE GEOPHYSICS LABORATORY
AIR FORCE SYSTEMS COMMAND
UNITED STATES AIR FORCE
HANSCOM AFB, MASSACHUSETTS 01731

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER AFGL-TR-78-0235	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) AN INVESTIGATION OF TWO MODELS FOR THE DEGREE VARIANCES OF GLOBAL COVARIANCE FUNCTIONS		5. TYPE OF REPORT & PERIOD COVERED Scientific. Interim Scientific Report No. 18
		6. PERFORMING ORG. REPORT NUMBER Dept. of Geod. Sci. No. 275
7. AUTHOR(s) Christopher Jekeli		8. CONTRACT OR GRANT NUMBER(s) F19628-76-C-0010
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Geodetic Science The Ohio State University - 1958 Neil Avenue Columbus, Ohio 43210		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 62101F 760003 AG
11. CONTROLLING OFFICE NAME AND ADDRESS Air Force Geophysics Laboratory Hanscom AFB, Massachusetts 01731 Contract Monitor: Bela Szabo/LW		12. REPORT DATE September 1978
		13. NUMBER OF PAGES 78
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

A - Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Gravity, geodesy, anomaly covariance functions, potential.

E squared

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Degree variances of the covariance function for gravity anomalies are represented according to the two-component model suggested by Moritz in an attempt to arrive at a low horizontal gradient variance (200 E^2). The parameters of the model are determined through a least squares adjustment to data consisting of GEM 9 anomaly degree variances, as well as empirical point and mean anomaly variances. → next page

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Results show that this model can accommodate the given point and mean anomaly variances and degree variances, while also yielding the desired low gradient variance. A comparison of this and a similar one-component model investigated by Tscherning and Rapp indicates that the latter cannot produce a low gradient variance together with a satisfactory fit to the data. Also, it does not adapt as well to the observed (GEM 9) attenuation of the anomaly degree variances.

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Foreword

This report was prepared by Mr. Christopher Jekeli, Graduate Research Associate, Department of Geodetic Science, The Ohio State University, under Air Force Contract No. F19628-76-C-0010, The Ohio State University Research Foundation Project No. 710335, Project Supervisor, Richard H. Rapp. The contract covering this research is administered by the Air Force Geophysics Laboratory, L. G. Hanscom Air Force Base, Massachusetts, with Mr. Bela Szabo, Contract Monitor.

Acknowledgements

I wish to extend my sincerest gratitude to Professor Richard H. Rapp who as my advisor proposed this study and who never failed to provide guidance and helpful advice in all areas of the work.

Special thanks go to Dr. Helmut Moritz who was able to take the time to serve on the reading committee for the thesis examination. His thought provoking remarks were particularly appreciated.

Several constructive comments by Mr. Christian C. Tscherning were also instrumental in clarifying key aspects of the theory. For this, I am truly thankful.

Finally, I am indebted to Pamela Pozderac for her conscientious and expeditious typing of the completed draft.

Table of Contents

I. Introduction

The gravity field of the earth has acquired fundamental significance in geodesy. All geodetic measurements, except those of distance, directly depend on this field. For example, the measurements of horizontal angles and zenith angles are made with respect to the direction of the plumb line; and heights are referenced to the geoid. In precise geodesy, the small differences between the actual gravity and normal gravity, as defined by an equipotential ellipsoid, must be taken rigorously into account.

The most common, directly observable, gravimetric quantity is the gravity anomaly. It can be related, in theory and with certain approximations, to the remaining elements of the anomalous gravity field, such as potential harmonic coefficients, or to the deflections of the vertical (Vening Meinesz formula) and to geoid undulations (Stokes' formula) (Heiskanen and Moritz 1967). Unfortunately, the evaluations of these types of integral equations presupposes that the anomalies are known everywhere on the earth's surface. Since, at present, this is not the case, one attempts to estimate (predict, interpolate, or extrapolate) gravity anomalies at the unsurveyed points.

The methods for such determinations include collocation or least squares prediction (Moritz 1972). This type of estimation of a "signal" quantity of the earth's gravity field, such as gravity anomalies, requires either a local or a global covariance function; the choice depends on the particular problem. The covariance function characterizes the statistical or random nature of gravity anomalies. Therefore, it would be perfectly determined if gravity anomalies were known over the entire surface of the earth. In the absence of this knowledge, it is often replaced by an analytical (smooth) function, which is judged to agree well with the behavior of known gravity data. The fact that one can work only with an approximation to the true covariance function does not invalidate the method of collocation. It simply means that the resulting predictions are not optimal in the sense of having the least error (Moritz 1976). The example above of a covariance function is more precisely an autocovariance function for gravity anomalies; a crosscovariance function describes the covariance between different signal or random quantities.

The principal objective of this study is to determine the parameters of a model for the global covariance function of gravity anomalies. The information that is available for such a determination consists, in this case, of a finite set of anomaly degree variances, as well as mean and point anomaly variances, all estimated from satellite data and terrestrial gravimetry. Essentially, the investigations herein expand some of the ideas and computations presented by Moritz (1976, 1977).

Throughout all derivations, the earth is assumed to be a sphere having some mean radius R. This implies that the atmosphere, the earth's ellipticity, and the terrain are ignored. For practical considerations, however, one might make the distinction between the Bjerhammar sphere (which is a sphere entirely enclosed within the earth, Krarup (1969)) and the mean earth sphere with radius $R_e = 6371$ km. The former would be desirable in order to apply the formulas in practice, at or near the earth's surface, with no risk of nonconvergence of the series in powers of $\frac{R}{r}$ (r being the distance from the earth's center to the computation point). To be sure, a series such as for the potential is guaranteed to converge only exterior to the sphere that encloses all terrestrial masses. The problem of formal convergence above and close to the earth's surface is not considered here; for a treatment of the difficulties that arise and the approximations that can be made, one might refer to Sjöberg (1977).

In formulating the many equations, it is attempted to maintain the most convenient notation, while also adhering as much as possible to the conventional and adopted symbols of the principal references.

Although no confusion is anticipated, the following distinctions are emphasized. Within this text, R denotes a fixed, but arbitrary radius (e.g. $R = R_B$ = Bjerhammar sphere radius); R_e represents the (fixed) radius of the mean earth, it is not arbitrary ($R_e = 6371$ km); R_c refers to the radius of a variable sphere ($R_c \geq R$); while r denotes the usual variable coordinate in the system of spherical coordinates.

The next section reviews the theoretical background of covariance functions and develops the interrelationships between the covariance functions of the disturbing potential, the gravity anomaly, the mean gravity anomaly, and the radial derivative of the gravity anomaly. The third section presents the model for the covariance function and the computational procedures that are undertaken to determine the corresponding parameters. Basically, they are found by fitting the model to the given data along the ideas of a least squares adjustment. The results of these procedures are discussed in the fourth section, culminating in a comparison of the models of Tscherning and Rapp (1974) and Moritz (1977).

II. Theoretical Aspects

The underlying assumption is that the anomalous potential (or the gravity anomaly field) represents a two-dimensional, stationary stochastic process on the sphere of the earth. The usual definition of a stochastic (or random) process (Papoulis 1965) is that it is a set of functions, each assigned to an outcome of an experiment and depending on time. For a fixed instant in time,

the function is a random variable. In applying these notions to the gravity field, the (one-dimensional) time is replaced by the two-dimensional surface of a sphere (earth). The process is considered to be stationary, if it has no point of origin in time, or in the present case, on the sphere (homogeneity). One further presupposes a condition of isotropy, implying complete rotational symmetry of the sphere, or equivalently, independence of direction. The concept of a stochastic process is probabilistic in nature which may, or may not, conflict with one's perception of the gravity field. For some arguments and discussions on the validity of the assumption above, one can consult, for example, Moritz (1972) and Meissl (1971). Nevertheless, the application of a stationary, isotropic, stochastic process to the gravity field represents an approximation whose degree of accuracy on a global basis may be quite good; while locally, the properties of stationarity and isotropy may have only debatable justification.

An alternate point of view that the gravity field should be treated as a deterministic phenomenon is advocated, for example, by Krarup (1969). It is not the intent here to expound on these philosophical questions, and we adopt the (perhaps more easily understood) ideas presented explicitly by Moritz (1972).

II.1 Definition of Covariance Function

In adopting the statistical approach, the average, or mean over the unit sphere is defined by

$$M(\cdot) = \frac{1}{4\pi} \iint_{\sigma} (\cdot)\, d\sigma \tag{2.1}$$

where $d\sigma$ denotes an element of area on the unit sphere σ. Let $T(r_P, \theta_P, \lambda_P)$ and $T(r_Q, \theta_Q, \lambda_Q)$ be the values of the disturbing potential at points P and Q, respectively. r, θ, λ are the usual spherical coordinates (θ = colatitude, λ = longitude). Then the covariance between these potentials on the unit sphere is given by (set $r_P = 1 = r_Q$)

$$\begin{aligned} \mathrm{cov}(T_P, T_Q) &= M[(T_P - M(T_P))(T_Q - M(T_Q))] \\ &= M(T_P T_Q) - M(T_P)\, M(T_Q) \end{aligned} \tag{2.2}$$

We assume that the signal field, i.e. the disturbing potential, averages to zero over the sphere, which means that the spherical harmonic expansion of T does not contain a zero-order (constant) term. More concretely, this is implied by the assumption that the masses of the earth and reference ellipsoid are equal (Heiskanen and Moritz 1967, p. 98). Thus

$$M(T_P) = M(T_Q) = 0 \tag{2.3}$$

and

$$\mathrm{cov}(T_P, T_Q) \equiv K(P,Q) = M(T_P T_Q) \tag{2.4}$$

According to the property of homogeneity, the covariance depends only on the relative positions of the points, assumed now to be on the sphere. With the additional stipulation of isotropy, the dependence is only on the spherical distance ψ between P and Q. Therefore, $M(T_P T_Q)$ is a function of one variable ψ and represents the average, over the sphere, of all possible products $T_P T_Q$ for points separated by the distance ψ:

$$M(T_P T_Q) = \frac{1}{8\pi^2} \int_{\alpha=0}^{2\pi} \int_{\lambda_P=0}^{2\pi} \int_{\theta_P=0}^{\pi} T(1, \theta_P, \lambda_P)\, T(1, \theta_Q, \lambda_Q) \sin\theta_P \, d\theta_P \, d\lambda_P \, d\alpha \tag{2.5}$$

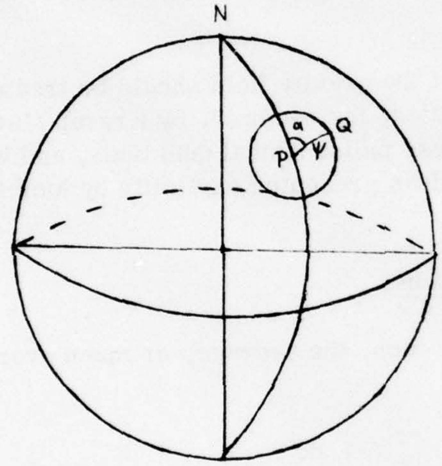


Figure 2.1

The first integral (α = azimuth) denotes the average over a circle centered at P with radius ψ, while the last two integrals represent the average of all such circles over the sphere. The azimuth α is related to the coordinates of P and Q as follows (consider the polar triangle NPQ):

$$\tan\alpha = \frac{\sin\theta_Q \sin(\lambda_Q - \lambda_P)}{\sin\theta_P \cos\theta_Q - \cos\theta_P \sin\theta_Q \cos(\lambda_Q - \lambda_P)} \tag{2.6}$$

One could as well have written $M(T_P T_Q)$ in terms of an average with respect to circles centered at points Q ($M(T_P T_Q) = M(T_Q T_P)$):

$$M(T_P T_Q) = \frac{1}{8\pi^2} \int_{\alpha=0}^{2\pi} \int_{\lambda_Q=0}^{2\pi} \int_{\theta_Q=0}^{\pi} T_P T_Q \sin\theta_Q \, d\theta_Q \, d\lambda_Q \, d\alpha \tag{2.7}$$

where now α is defined analogously to (2.6), but with the indices P and Q interchanged.

Since the potential is harmonic in space, the following Laplace equations are satisfied:

$$\Delta_P T_P = 0 \quad , \quad \Delta_Q T_Q = 0 \tag{2.8}$$

where the Laplacian operator in spherical coordinates is

$$\Delta_k = \frac{\partial^2}{\partial r_k^2} + \frac{2}{r_k}\frac{\partial}{\partial r_k} + \frac{1}{r_k^2}\frac{\partial^2}{\partial \theta_k^2} + \frac{\cot\theta_k}{r_k^2}\frac{\partial}{\partial\theta_k} + \frac{1}{r_k^2\sin^2\theta_k}\frac{\partial^2}{\partial\lambda_k^2} \; , \; k = P, Q \tag{2.9}$$

It is evident that there does not exist another equation, independent of (2.6), which relates all three quantities α, point P, and point Q. That is, two of these are independent. Suppose that this is the case for α and point Q. Upon applying the operator Δ_Q to $M(T_P T_Q)$ as given by (2.5) and extended by reinstating r_P and r_Q ($M(T_P T_Q)$ then depends in some manner on r_P, r_Q), it can be taken inside the integrals, thereby giving

$$\Delta_Q M(T_P T_Q) = \frac{1}{8\pi^2}\int_\alpha\int_{\lambda_P}\int_{\theta_P} T_P \, \Delta_Q T_Q \sin\theta_P \, d\theta_P \, d\lambda_P \, d\alpha = 0 \tag{2.10}$$

Similarly, if one regards α and point P as being independent, then in using (2.7), we have

$$\Delta_P M(T_P T_Q) = \frac{1}{8\pi^2}\int_\alpha\int_{\lambda_Q}\int_{\theta_Q} T_Q \, \Delta_P T_P \sin\theta_Q \, d\theta_Q \, d\lambda_Q \, d\alpha = 0 \tag{2.11}$$

It is hereby shown that the anomalous potential covariance function, K(P, Q), as a function of P only is harmonic in P; and as a function of Q only, it is harmonic in Q (in the same regions where T is harmonic).

A further property of a covariance function, which is important when formulating the type of models that are of interest here, is the property of positive definiteness. The fact that K (P, Q) (or any covariance function) is positive definite is proved by Moritz (1976, p. 12). Briefly, given any linear combination of quantities in the signal field, for example

$$T = \sum_{i=1}^{n} \mu_i T_i \; , \tag{2.12}$$

the average of its square should be positive:

$$0 < M(T^2) = \sum_{i=1}^{n} \sum_{j=1}^{n} M(T_i T_j)\, \mu_i \mu_j \tag{2.13}$$

This is precisely the definition of positive definiteness for $M(T_i T_j)$.

II. 2 Series Expansion of Covariance Functions

Let $K(P, Q)$ be the covariance function of the anomalous potential T with arguments P, Q representing, in general, two points on or exterior to the earth. For the moment, let point P be fixed on the unit sphere and let Q vary on this sphere. Without loss in generality, one may shift the earth's pole to P (see Figure 2.1) and introduce a new spherical coordinate system ψ, α, where ψ is the spherical distance from P to Q and α is the azimuth of Q at P (with respect to the meridian through P). Then, $K(P,Q)$ being a harmonic function in Q can be expanded in this coordinate system as a series of spherical harmonic functions:

$$K(P,Q) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} (k_{nm} \cos m\alpha + \ell_{nm} \sin m\alpha)\, P_{nm}(\cos \psi) \tag{2.14}$$

P_{nm} denotes the associated Legendre function and k_{nm}, l_{nm} are harmonic coefficients, Invoking the property of isotropy (no dependence on α), it is immediately recognized that one may write

$$K(P,Q) = \sum_{n=0}^{\infty} k_n P_n(\cos \psi) \tag{2.15}$$

where $k_{n0} \equiv k_n$, and $P_{n0} \equiv P_n$ is the Legendre polynomial. The k_n viewed as Fourier coefficients are obtained in theory from

$$k_n = \frac{2n+1}{2} \int_0^{\pi} K(P,Q)\, P_n(\cos \psi) \sin \psi \, d\psi \tag{2.16}$$

(P and Q are on the unit sphere). Substituting the definition of $K(P,Q)$, (2.5) into (2.16), we can readily show (cf. Heiskanen and Moritz 1967, sec. 7-3) that also

$$k_n = \sum_{m=0}^{n} (\overline{A}_{nm}^2 + \overline{B}_{nm}^2) \tag{2.17}$$

where on this sphere

$$T = \sum_{n=0}^{\infty} T_n \tag{2.18}$$

and the surface harmonics are given by

$$T_n = \sum_{m=0}^{n} (\overline{A}_{nm} \cos m\lambda + \overline{B}_{nm} \sin m\lambda)\, \overline{P}_{nm}(\cos\theta) \tag{2.19}$$

Here, $\overline{A}_{nm}$, $\overline{B}_{nm}$ are fully normalized harmonic coefficients, $\overline{P}_{nm}$ is a fully normalized associated Legendre function, and θ, λ again are geocentric spherical coordinates (Heiskanen and Moritz 1967).

Now let P, Q vary in the space outside the earth, then K (P, Q) must depend in some way on the coordinates r_P and r_Q. In fact, it is well known, that the dependence of a harmonic function on r is of the form $r^{-(n+1)}$ (that is, when $1 \le r < \infty$). Since K (P, Q) is harmonic in both P and Q, the spatial extension of equation (2.15) is

$$K(P,Q) = \sum_{n=0}^{\infty} k_n \left(\frac{R^2}{r_P\, r_Q}\right)^{n+1} P_n(\cos\psi) \tag{2.20}$$

where the unit sphere has now been replaced by a sphere of radius R; this series converges on or outside this sphere: for r_P, $r_Q \ge R$.

If $\Delta g(r,\theta,\lambda)$ denotes the gravity anomaly function, then $r\Delta g(r,\theta,\lambda)$ is a harmonic function (Heiskanen and Moritz 1967, p. 90). The same analysis may be applied to it as for $T(r,\theta,\lambda)$ to obtain the spatial covariance function for the gravity anomalies:

$$C(P,Q) = \sum_{n=0}^{\infty} c_n \left(\frac{R^2}{r_P\, r_Q}\right)^{n+2} P_n(\cos\psi) \tag{2.21}$$

where the c_n are the corresponding Fourier coefficients, commonly known as anomaly degree variances.

The following sections elaborate on these covariance functions, as well as on those for the geoid undulation, the vertical gradient of gravity anomalies, and the mean gravity anomaly; and several interrelationships are also derived.

II. 3 The Anomalous Potential Covariance Function

From the discussion by Heiskanen and Moritz (1967, pp. 107-108), the anomalous potential can be obtained as

$$T(r,\theta,\lambda) = \frac{kM}{r}\sum_{n=2}^{\infty}\left(\frac{R}{r}\right)^{n}\sum_{m=0}^{n}(\bar{C}_{nm}\cos m\lambda + \bar{S}_{nm}\sin m\lambda)\,\bar{P}_{nm}(\cos\theta) \tag{2.22}$$

Here, kM is the product of the gravitational constant and the mass of the earth; R is the radius of the sphere to which the $\bar{C}_{nm}$ and $\bar{S}_{nm}$ refer. The latter are appropriately defined potential coefficients (accounting for the removal of the reference field). Setting r = R yields the anomalous potential on the sphere of radius R:

$$T = \sum_{n=2}^{\infty} T_n \tag{2.23}$$

with

$$T_n(\theta,\lambda) = \frac{kM}{R}\sum_{m=0}^{n}(\bar{C}_{nm}\cos m\lambda + \bar{S}_{nm}\sin m\lambda)\,\bar{P}_{nm}(\cos\theta) \tag{2.24}$$

($T_0 = T_1 = 0$, requiring that the masses of the earth and reference ellipsoid are equal, and that the center of the ellipsoid is located at the earth's center of mass). Also, by setting $r_P = r_Q = R$ in equation (2.20), the covariance function for T on the sphere of radius R is

$$K(\psi) = \sum_{n=2}^{\infty} k_n P_n(\cos\psi) \tag{2.25}$$

where now (in view of equations (2.24) and (2.17))

$$k_n = \sum_{m=0}^{n}\left(\frac{kM}{R_c}\right)^2(\bar{C}_{nm}^2 + \bar{S}_{nm}^2) \tag{2.26}$$

On the sphere of radius R_c ($\geq R$), we obtain

$$k_n(R_c) = \sum_{m=0}^{n}\left(\frac{kM}{R}\right)^2\left(\frac{R^2}{R_c^2}\right)^n(\bar{C}_{nm}^2 + \bar{S}_{nm}^2) \tag{2.27}$$

by setting $r = R_c$ in (2.22). This shows that $k_n = \left(\frac{R_c^2}{R^2}\right)^{n+1} k_n(R_c)$, and hence

the spatial covariance function (2.20) can be written equivalently as

$$K(P,Q) = \sum_{n=0}^{\infty} k_n(R_C) \left(\frac{R_C^2}{r_P r_Q}\right)^{n+1} P_n(\cos\psi) \tag{2.28}$$

II.4 The Gravity Anomaly Covariance Function

The gravity anomaly function Δg is given by (Heiskanen and Moritz 1967, p. 89):

$$\Delta g(r,\theta,\lambda) = \frac{1}{r}\sum_{n=2}^{\infty} (n-1)\left(\frac{R}{r}\right)^{n+1} T_n(\theta,\lambda) \tag{2.29}$$

T_n is evaluated on the sphere of radius R. Loosely stated, the series converges outside this sphere. As a reminder, when applied to the real world, the convergence depends on the selection of R and on the values of the harmonic coefficients C_{nm}, S_{nm}. Again, any ensuing difficulties are ignored here (see section I). Substituting equation (2.24), one gets

$$\Delta g(r,\theta,\lambda) = \frac{kM}{R^2}\sum_{n=2}^{\infty} (n-1)\left(\frac{R}{r}\right)^{n+2} \sum_{m=0}^{n} (\bar{C}_{nm}\cos m\lambda + \bar{S}_{nm}\sin m\lambda)\,\bar{P}_{nm}(\cos\theta) \tag{2.30}$$

Suppose that r is fixed, say $r = R_C \geq R$, then

$$\Delta g(R_C,\theta,\lambda) = \frac{kM}{R^2}\sum_{n=2}^{\infty} (n-1)\left(\frac{R}{R_C}\right)^{n+2} \sum_{m=0}^{n} (\bar{C}_{nm}\cos m\lambda + \bar{S}_{nm}\sin m\lambda)\,\bar{P}_{nm}(\cos\theta) \tag{2.31}$$

are gravity anomalies on the sphere of radius R_C. That is, if Δg is expanded in surface harmonics Δg_n on this sphere, then they would be given by

$$\Delta g_n(\theta,\lambda) = \frac{kM}{R^2}(n-1)\left(\frac{R}{R_C}\right)^{n+2} \sum_{m=0}^{n} (\bar{C}_{nm}\cos m\lambda + \bar{S}_{nm}\sin m\lambda)\,\bar{P}_{nm}(\cos\theta) \tag{2.32}$$

so that

$$\Delta g(R_C,\theta,\lambda) = \sum_{n=}^{\infty} \Delta g_n(\theta,\lambda) \tag{2.33}$$

The harmonic coefficients are thus

$$(\bar{a}_{nm}, \bar{b}_{nm}) = \frac{kM}{R^2} (n-1) \left(\frac{R}{R_c}\right)^{n+2} (\bar{C}_{nm}, \bar{S}_{nm}) \tag{2.34}$$

where we have written

$$\Delta g_n(\theta, \lambda) = \sum_{m=0}^{n} (\bar{a}_{nm} \cos m\lambda + \bar{b}_{nm} \sin m\lambda) \bar{P}_{nm}(\cos \theta) \tag{2.35}$$

Going back to equation (2.21), one may consider the covariance function

$$C(P, Q) = \sum_{n=2}^{\infty} c_n(R_c) \left(\frac{R_c^2}{r_P r_Q}\right)^{n+2} P_n(\cos \psi) \tag{2.36}$$

in which $c_n(R_c)$ now refers to the sphere of radius R_c. (That is to say,

$$c_n(R_c) = \frac{2n+1}{2} \int_0^{\pi} C(P, Q) \bigg|_{\substack{r_P = R_c \\ r_Q = R_c}} P_n(\cos \psi) \sin \psi \, d\psi \tag{2.37}$$

cf. equation (2.16).) In analogy to equation (2.17), the anomaly degree variances are then given by

$$c_n(R_c) = \sum_{m=0}^{n} (\bar{a}_{nm}^2 + \bar{b}_{nm}^2) = \left(\frac{kM}{R^2}\right)^2 (n-1)^2 \left(\frac{R^2}{R_c^2}\right)^{n+2} \sum_{m=0}^{n} (\bar{C}_{nm}^2 + \bar{S}_{nm}^2) \tag{2.38}$$

Setting $\gamma = \frac{kM}{R^2}$ as an average value of gravity (on the sphere of radius R),

$$c_n(R_c) = \gamma^2 (n-1)^2 \left(\frac{R^2}{R_c^2}\right)^{n+2} \sum_{m=0}^{n} (\bar{C}_{nm}^2 + \bar{S}_{nm}^2) \tag{2.39}$$

Comparing (2.27) and (2.39), it is seen that

$$c_n(R_c) = (n-1)^2 \frac{1}{R_c^2} k_n(R_c) \tag{2.40}$$

If $R = R_B$ is the radius of the Bjerhammar sphere (see section I), and if $R_c = R = R_B$, then the c_n as computed from (2.39):

$$c_n(R_B) \equiv c_n' = \gamma^2 (n-1)^2 \sum_{m=0}^{n} (\bar{C}_{nm}^2 + \bar{S}_{nm}^2) \tag{2.41}$$

are the degree variances which refer to the Bjerhammar sphere.

Finally, it is noted that the anomaly variance on the sphere of radius R_c is defined by

$$C(P, P) = \sum_{n=2}^{\infty} c_n(R_c)\, P_n(\cos 0) \tag{2.42}$$

or

$$C_0 = \sum_{n=2}^{\infty} c_n(R_c)$$

where $c_n(R_c)$ refers to this sphere.

II.5 Undulation Covariance Function

Turning now to geoid undulations, Bruns' formula states that

$$N(R_c, \theta, \lambda) = \frac{1}{\gamma_c}\, T(R_c, \theta, \lambda) \tag{2.43}$$

where $\gamma_c = \dfrac{kM}{R_c^2}$ is normal gravity assumed constant on the sphere of radius R_c (spherical approximation), and N (to a first approximation) is the undulation; when $R_c = R_e$, it is the separation between the geoid and the ellipsoid of the same potential. (Of course, it is also assumed that there is no mass outside the geoid.) By setting $r = R_c$ in equation (2.22) for the disturbing potential, one obtains the undulation as a function on the sphere of radius R_c:

$$N(R_c, \theta, \lambda) = R \sum_{n=2}^{\infty} \left(\frac{R}{R_c}\right)^{n-1} \sum_{m=0}^{n} (\overline{C}_{nm} \cos m\lambda + \overline{S}_{nm} \sin m\lambda)\, \overline{P}_{nm}(\cos\theta) \tag{2.44}$$

In general, the spatial covariance function for N can be written as

$$L(P, Q) = \sum_{n=2}^{\infty} \ell_n \left(\frac{R^2}{r_P\, r_Q}\right)^{n-1} P_n(\cos\psi), \quad \ell_n = \frac{R^4}{(kM)^2}\, k_n \tag{2.45}$$

since $r^{-2} N$ is proportional to T (cf. equ. (2.20) for T). Above, the coefficients l_n refer to a sphere of radius R. We will be interested primarily in the undulation variance, L_0, on the sphere of radius R_c. This is obtained by setting $\psi = 0$, i.e. $P = Q$, and $r_P = r_Q = R_c$ in (2.45):

$$L_0 = \sum_{n=2}^{\infty} \ell_n \left(\frac{R^2}{R_c^2}\right)^{n-1} = \sum_{n=2}^{\infty} \ell_n(R_c) \tag{2.46}$$

where $\ell_n(R_c)$ refers to the sphere of radius R_c. Again, by analogy to equation

(2.17) and in view of (2.44), these undulation degree variances can be expressed as

$$\ell_n(R_C) = R^2 \left(\frac{R^2}{R_C^2}\right)^{n-1} \sum_{m=0}^{n} (\bar{C}_{nm}^2 + \bar{S}_{nm}^2) \tag{2.47}$$

and using (2.39) with $\gamma_C = \frac{kM}{R_C^2}$, this reduces to

$$\ell_n(R_C) = \frac{R_C^2}{(n-1)^2} \frac{1}{\gamma_C^2} c_n(R_C) \tag{2.48}$$

so that on the sphere of radius R_C,

$$L_0 = \sum_{n=2}^{\infty} \frac{R_C^2}{(n-1)^2} \frac{1}{\gamma_C^2} c_n(R_C) \tag{2.49}$$

II.6 Mean Anomaly Covariance Function

The value of the gravity anomaly function on the sphere of radius R_C (equation (2.31)),

$$\Delta g(\theta, \lambda) = \frac{kM}{R^2} \sum_{n=2}^{\infty} (n-1) \left(\frac{R}{R_C}\right)^{n+2} \sum_{m=0}^{n} (\bar{C}_{nm} \cos m\lambda + \bar{S}_{nm} \sin m\lambda) \bar{P}_{nm}(\cos\theta) \tag{2.50}$$

is generally different for each point on this sphere. Consequently, its covariance function (2.36) yields covariances between point gravity anomalies. In practice when carrying out gravity measurements over large areas, one must often be content with only mean, or representative values of the anomalies. To obtain the corresponding covariance function, we first find a functional formulation of mean gravity anomalies. For instance, one can apply an averaging operator to equation (2.50) over a spherical cap of radius ψ_0:

$$\overline{\Delta g}(\bar{\theta}, \bar{\lambda}) = \iint_{\sigma} A(\psi) \, \Delta g(\theta, \lambda) \, d\sigma \tag{2.51}$$

Here, for generality, σ is the entire surface of the sphere (see the definition of $A(\psi)$ below); ψ is the spherical distance from $\bar{\theta}, \bar{\lambda}$ to θ, λ; $d\sigma = \sin\theta \, d\theta d\lambda$; and $(\bar{\theta}, \bar{\lambda})$ is a pair of coordinates within the cap (e.g. at its center). The kernel $A(\psi)$ of this integral operator can be defined over the sphere by

$$A(\psi) = \begin{cases} \dfrac{1}{2\pi} \dfrac{1}{1-\cos\psi_0}, & \text{for } \psi \le \psi_0 \\ 0, & \text{for } \psi > \psi_0 \end{cases} \tag{2.52}$$

That is, it is the reciprocal of the area of the spherical cap:

$$\int_{\alpha=0}^{2\pi} \int_{\psi=0}^{\psi_0} \sin\psi \, d\psi \, d\alpha = 2\pi(1-\cos\psi_0) \tag{2.53}$$

so that $\overline{\Delta g}(\overline{\theta}, \overline{\lambda})$ in (2.51) is the average gravity anomaly over the spherical cap σ_c :

$$\overline{\Delta g}(\overline{\theta}, \overline{\lambda}) = \frac{1}{2\pi(1-\cos\psi_0)} \iint_{\sigma_c} \Delta g(\theta, \lambda) \, d\sigma \tag{2.54}$$

The kernel of the smoothing operator is more generally viewed as a weight function, of which the simple average defined by (2.52) is a special case (Pellinen 1966). It is shown by Meissl (1971) that the eigenfunctions of isotropic operators, exemplified by the integral operator above, are the spherical harmonics $\overline{P}_{nm}(\cos\theta)\cos m\lambda$ and $\overline{P}_{nm}(\cos\theta)\sin m\lambda$:

$$\begin{aligned} \iint_{\sigma} A(\psi)\cos m\lambda \, \overline{P}_{nm}(\cos\theta) \, d\sigma &= \beta_n \cos m\overline{\lambda} \, \overline{P}_{nm}(\cos\overline{\theta}) \\ \iint_{\sigma} A(\psi)\sin m\lambda \, \overline{P}_{nm}(\cos\theta) \, d\sigma &= \beta_n \sin m\overline{\lambda} \, \overline{P}_{nm}(\cos\overline{\theta}) \end{aligned} \tag{2.55}$$

where $\cos\psi$ is a function of θ, λ, $\overline{\theta}$, $\overline{\lambda}$ (the integration is with respect to θ, λ). The β_n's are the corresponding eigenvalues of the operator (depending only on the degree n). It is further shown by Meissl (1971) that one can evaluate the eigenvalues according to the so-called Funk-Hecke formula

$$\beta_n = 2\pi \int_{-1}^{1} A(t) \, P_n(t) \, dt \tag{2.56}$$

in which $t = \cos\psi$ and P_n is the n^{th}-degree Legendre polynomial. Putting the definition of $A(\psi)$ into (2.56), we obtain

$$\beta_n = \frac{1}{1-\cos\psi_0} \int_{\cos\psi_0}^{1} P_n(t) \, dt \tag{2.57}$$

and invoking the recursion formula

$$\frac{d}{dt} P_{n+1}(t) - \frac{d}{dt} P_{n-1}(t) = (2n+1) P_n(t) \tag{2.58}$$

this simplifies to

$$\beta_n = \frac{1}{2n+1} \frac{1}{1-\cos\psi_0} [\, P_{n-1}(\cos\psi_0) - P_{n+1}(\cos\psi_0) \,] \qquad (2.59)$$

Now, putting (2.50) into (2.51), and recalling (2.55), one can write

$$\overline{\Delta g}(\overline{\theta},\overline{\lambda}) = \frac{kM}{R^2} \sum_{n=2}^{\infty} (n-1) \left(\frac{R}{R_c}\right)^{n+2} \beta_n \sum_{m=0}^{n} (\overline{C}_{nm} \cos m\overline{\lambda} + \overline{S}_{nm} \sin m\overline{\lambda}) \overline{P}_{nm}(\cos\overline{\theta}) \qquad (2.60)$$

which is the average gravity anomaly over the cap σ_c on the sphere of radius R_c. If one specifies that $\overline{\theta}, \overline{\lambda}$ are always the coordinates of the center of a cap, then $\overline{\Delta g}(\overline{\theta},\overline{\lambda})$ as given by (2.60) is a well-defined, continuous function over the sphere and may be regarded as a smoothed gravity anomaly function with the smoothing factors β_n. That is, the irregularities of $\Delta g(\theta,\lambda)$ have been smoothed (averaged) out.

It is now readily seen (for instance, by applying the law of propagation of covariances, see Moritz (1972) and p. 15) that the degree variances of the mean anomaly covariance function $\overline{C}(P,Q)$ are

$$\overline{c}_n = \beta_n^2 c_n \qquad (2.61)$$

and that as in (2.36),

$$\overline{C}(P,Q) = \sum_{n=2}^{\infty} \beta_n^2 c_n s^{n+2} P_n(\cos\psi) \qquad (2.62)$$

We have set $s = \frac{R_c^2}{r_P r_Q}$ and P, Q are the centers of the respective spherical caps. As written here, $\overline{C}(P,Q)$ is the mean anomaly covariance function referring to the sphere of radius R_c. The mean anomaly variance on this sphere is found by setting $\psi = 0$ and $r_P = r_Q = R_c$:

$$C_0 = \sum_{n=2}^{\infty} \beta_n^2 c_n \qquad (2.63)$$

One can now restrict the domain of the function (2.62) such that the coordinates of the two points P, Q refer to the centers of two disjoint spherical caps. In practice, the spherical caps are often approximated by blocks of sizes such as 1° x 1° or 5° x 5°, and the radius ψ_0 is chosen such that the area of the cap equals the area of the block.

II.7 Vertical Gradient Covariance Function

The final covariance function to be considered is the one for the vertical "gradient" of the gravity anomaly, i.e. for

$$\frac{\partial}{\partial r} \Delta g(r,\theta,\lambda) \qquad (2.64)$$

The covariance function for the gravity anomalies (equation (2.21)) is repeated here:

$$C(P,Q) = \sum_{n=2}^{\infty} c_n \left(\frac{R^2}{r_P\, r_Q}\right)^{n+2} P_n(\cos\psi) \tag{2.65}$$

Using (2.64) and the harmonic expansion of $\Delta g(r,\theta,\lambda)$, it is possible now to develop, as before, the relationship between the covariance functions of the vertical gradient and the gravity anomaly. However, for the sake of variety, we employ the law of propagation of covariances (Moritz 1972, p. 97), which in the present situation is formulated as

$$G(P,Q) = \frac{\partial}{\partial r_P}\left(\frac{\partial}{\partial r_Q} C(P,Q)\right) \tag{2.66}$$

where G(P, Q) is the covariance function of the vertical gradient. Using equation (2.65), the calculations of (2.66) lead to

$$\begin{aligned} G(P,Q) &= \frac{\partial}{\partial r_P}\left[-\sum_{n=2}^{\infty} c_n \frac{n+2}{r_Q}\left(\frac{R^2}{r_P\, r_Q}\right)^{n+2} P_n(\cos\psi)\right] \\ &= \sum_{n=2}^{\infty} c_n \frac{(n+2)^2}{R^2}\left(\frac{R^2}{r_P\, r_Q}\right)^{n+3} P_n(\cos\psi) \end{aligned} \tag{2.67}$$

If instead, c_n is chosen to refer to a sphere of radius R_c, similar manipulations show that

$$G(P,Q) = \sum_{n=2}^{\infty} c_n(R_c)\frac{(n+2)^2}{R_c^2}\left(\frac{R_c^2}{r_P\, r_Q}\right)^{n+3} P_n(\cos\psi) \tag{2.68}$$

or

$$G(P,Q) = \sum_{n=2}^{\infty} g_n(R_c)\left(\frac{R_c^2}{r_P\, r_Q}\right)^{n+3} P_n(\cos\psi) \tag{2.69}$$

where

$$g_n(R_c) = \frac{(n+2)^2}{R_c^2} c_n(R_c) \tag{2.70}$$

are the gradient degree variances referring to the sphere of radius R_c.

From equations (2.68) and (2.69), one can immediately draw some conclusions as to the characteristics of the function G(P, Q). Considering the magnitude of c_n (see Table III), g_n is practically insignificant for small degree n, due to the divisor R_c^2. This implies that G (P, Q) is a very local function, depending mainly on the high-degree (short wavelength) variations in the gravity field. Therefore, given anomaly degree variances, say, computed from potential coefficients (equation (2.39)) for n = 2, . . . , 20, the approximation

$$G(P,Q) \approx \sum_{n=2}^{20} g_n \left(\frac{R_c^2}{r_P\, r_Q}\right)^{n+3} P_n(\cos\psi) \qquad (2.71)$$

computed through equation (2.70) is completely meaningless. In other words, low-degree information on the gravity field contributes little to the covariance function of the anomaly gradient. The consequence of this in the adjustment procedure (to be discussed later) is that one cannot solve for the (global) gradient variance G_0, which is defined (on the sphere of radius R_c) by

$$G_0 \equiv G(P,P) = \sum_{n=2}^{\infty} g_n \qquad (2.72)$$

when only c_n's of low degree are provided.

To further illustrate the local nature of this covariance function, several instances may be cited in the literature in which the authors have determined the anomaly gradient over specific areas which exhibit totally incomparable variation. We discuss here only the vertical gradient variance G_0 (or the horizontal gradient variance, G_{0H}, which is approximately $\frac{1}{2}G_0$ - see the appendix). At the turn of the century, R. Eötvös observed horizontal gradients in the area near the city of Arad in western Rumania (Selényi, 1953, p. 126). If x, y denote two mutually perpendicular directions (north-south and east-west) in the horizontal plane, then the root mean square (RMS) values of the anomalous gradients are

$$\begin{aligned} 20.42\ E &\rightarrow G_{0H} \approx 417.11\ E^2 \quad \text{in the x-direction} \\ 31.07\ E &\rightarrow G_{0H} \approx 965.26\ E^2 \quad \text{in the y-direction} \end{aligned} \qquad (2.73)$$

where $1\ E = 1$ Eötvös $= 10^{-9}\ s^{-2} = .1$ mgal/km. A normal gradient of 8.1 E was subtracted from the observed gradients in the north-south direction. It is based on the Bessel ellipsoid and Helmert's normal gravity formula (Helmert 1884). (The normal part in the y-direction is zero.) The surveyed area near Arad extends about 20' in longitude and 10' in latitude.

Mueller (1964) presents a map of fairly large extent, covering parts of Maine, New Brunswick and Nova Scotia, and depicting vertical gradient contours. By placing a rectangular grid over this map and extracting some 150 values, the square of the RMS value (after the reference field (3085.5 E) corresponding to the international ellipsoid and gravity formula has been subtracted) of the horizontal gradient is

$$\tfrac{1}{2}(33.3\ E^2) = 16.7\ E^2 \qquad (2.74)$$

with a range of values from -16.5 E to 34.5 E.

Finally, to elucidate the extremes that can occur, we refer to Hein (1977) who has computed the anomalous vertical gravity gradients based on the international gravity formula and displayed on a contour map of an area in the "Odenwald" (latitudinal extent: $55^\circ.04$ to $55^\circ.13$; longitudinal extent: $34^\circ.93$ to $35^\circ.02$ (E)). By similarly placing a rectangular grid over this map and reading off 153 values (the area around the large anomaly of -1250 E was omitted), the RMS value squared is found to be an enormous 12373 E^2, which implies a horizontal gradient variance of about 6200 E^2. The range of values is -325 E to +350 E.

Caution must be exercised when comparing these various approximate determinations of the local gradient variance. The anomalous gradient values mentioned above were not all derived using the same reference field, and the RMS values were not all computed over the same extent in area. Nevertheless, it becomes apparent that a global value of G_0 will not only be difficult to determine, but may not have any practical significance locally.

We also note that Schwarz (1976) assumes in his computations that

$$30\ E^2 \le G_{0H} \le 200\ E^2,$$

while Moritz (1976, 1977) adopts $G_{0H} = 200\ E^2$ as an example. Tscherning (1976) finds $G_{0H} \approx 3500\ E^2$ using the model (3.12) for the degree variances. Jordan (1978) develops a covariance model for gravity anomalies based on the model outlined in section III.2 with the parameter values of Table I, but modified to account for isostatic compensation. The variance of the vertical gradient for this model was found to be 833.98 E^2 (see Table 4 of Jordan (1978, p. 1819)). It is also shown very nicely by Jordan (in his Table 4) that the terrain and crust of the earth are overwhelmingly responsible (99%) for the variation in the vertical gradient (i.e., there are no sources deep within the earth). This again reveals the extremely local character of the vertical gradient covariance function.

A geometric interpretation of the horizontal gradient variance is briefly mentioned. When the covariance function is to be applied locally, it might be described, according to Moritz (1976), by three parameters. In the case of the gravity anomaly function, $C(\psi)$, they are

C_0, the anomaly variance ($C_0 = C(0)$)

ξ, the correlation length ($C(\xi) = \frac{1}{2} C_0$)

χ, the curvature parameter ($\chi = \varkappa \xi^2 / C_0$, where $\varkappa$ is the curvature of the covariance function at $\psi = 0$).

It can be shown (Moritz 1976), that with a planar approximation, the horizontal gradient variance G_{0H}, is related to the curvature parameter through

$$\chi = G_{0H} \xi^2 / C_0 \tag{2.75}$$

The theoretical considerations concerning the gradient covariance function are concluded by a short derivation of the equation

$$G_{0H} = \tfrac{1}{2} G_0 \quad \text{(planar approximation)} \tag{2.76}$$

which is presented in the appendix and employs several relations already derived by Moritz (1976).

III. Computational Procedures

Many more relationships among the covariances of quantities of the anomalous gravity field, besides the ones used here, can be obtained through the law of propagation of covariances (Moritz 1972). The corresponding computations have been made by Tscherning (1976) and Tscherning and Rapp (1974); thereby, demonstrating that it is necessary to find a model for only one of the covariance functions, or more specifically, for one set of degree variances.

III.1 Kaula's Rule

Through an analysis of a gravity field obtained from satellite observations and gravity measurements, Kaula (1963) postulated his well known "rule of thumb":

$$\sigma(\overline{C}_{nm}, \overline{S}_{nm}) = \frac{10^{-5}}{n^2} \tag{3.1}$$

This states that the root mean square variation of a harmonic coefficient of the earth's gravity field is inversely proportional to the square of its degree. Morrison (1971) also points out that in considering all available gravity data, it seems likely that the decay of the harmonic coefficients is not exponential, but much more slowly, for instance, on the order of a negative power of n.

Through the acquisition of more and better satellite data, Kaula's rule became less applicable to the higher-degree harmonics. A comprehensive comparison of various modifications and generalizations of this rule (as suggested by different authors) is presented by Rapp(1972). Instead of the root mean square variation, models of the degree variance were in fact investigated. If k_n is the anomalous potential degree variance, then by the definition of $\sigma(\overline{C}_{nm}, \overline{S}_{nm})$ and in view of equation (2.26), the relationship between these two quantities is

$$\sigma(\overline{C}_{nm}, \overline{S}_{nm}) = \sqrt{\frac{1}{2n+1}\sum_{m=0}^{n}(\overline{C}_{nm}^2 + \overline{S}_{nm}^2)} = \sqrt{\frac{k_n}{(2n+1)R^2\gamma^2}} \tag{3.2}$$

where $\gamma = \frac{kM}{R^2}$ is an average value of gravity on the sphere of radius R. It is assumed that with respect to the order m of the coefficients of fixed degree, the root mean square value is invariant (isotropy). k_n is further related to the gravity anomaly degree variance, c_n, through equation (2.40). Kaula's rule implies the following model for c_n:

$$c_n = \frac{\mu(n-1)^2(2n+1)}{n^4} \approx \frac{2\mu}{n+\frac{3}{2}} \tag{3.3}$$

in which μ is a constant. The anomaly degree variance model that was finally proposed by Rapp (1972) on the basis of satellite and terrestrial gravity data is of the form

$$c_n = \frac{\alpha (n - 1)}{(n - 2)(n + \beta + \epsilon n^2)} \tag{3.4}$$

where α, β, ϵ are suitable constants.

An additional consideration is appropriate here; namely, in regard to the desirability to find closed expressions for the infinite series of the covariance function. Moritz (1976) describes the properties of covariance functions that are implied by several such convenient models.

In summary, the types of models above and the model to be investigated later are empirical in nature, based primarily on the observed variation in the harmonic coefficients.

III. 2 The Degree Variance Model of Heller and Jordan (1975)

A completely different approach, developed recently by Heller and Jordan (1975), is an attempt to model the variation of the anomalous potential by introducing a "white noise" shell at a certain depth beneath the earth's surface. The derivation of the covariance function begins with the Poisson integral for the disturbing potential T (Heiskanen and Moritz, 1967, pp. 35, 238):

$$T(r, \theta, \lambda) = \frac{R(r^2 - R^2)}{4\pi} \iint_{\sigma} \frac{T}{\ell^3} \, d\sigma \tag{3.5}$$

T is given on the shell of radius R, and ℓ is the distance from the point (r, θ, λ) to the point (θ', λ') on the shell σ, over which the integration is performed; $d\sigma = \sin\theta' d\theta' d\lambda'$. Very briefly, the covariance function (due to an uncorrelated or "white noise" disturbing potential) is specified on the shell of radius R. By applying the law of propagation of covariances to this function according to (3.5) and by stipulating stationarity and isotropy (see section II.1), the covariance function for the disturbing potential in the exterior space is found to be

$$K(P,Q) = \frac{D^3 (2R_e - D)^3 (r_P^2 r_Q^2 - (R_e - D)^4) K_O}{(R_e^4 - (R_e - D)^4)((R_e - D)^4 + r_P^2 r_Q^2 - 2 r_P r_Q (R_e - D)^2 \cos\psi)^{3/2}} \tag{3.6}$$

In this formula, K_O is the variance of the disturbing potential at the surface of the earth (assumed to be a sphere) due to the shell at depth D; R_e is the mean radius of the earth; and ψ is the spherical distance on the earth between the two vectors r_P and r_Q.

There are two parameters in the model above, D and K_0. To obtain a better fit to actual gravity data, several uncorrelated shells are introduced at various depths. The covariance function is then

$$K(P,Q) = \sum_{m=1}^{M} K_m(P,Q) \qquad (3.7)$$

in which D_m, K_{0m}, m = 1, ..., M are the parameters (K_{0m} is the contribution to the surface variance of the disturbing potential, due to the shell at depth D_m).

Again, through the law of propagation of covariances, one can deduce the covariance functions of the gravity anomaly, the anomalous vertical gradient, etc. Jordan (1978) expands the covariance function for the disturbing potential into a series of Legendre polynomials:

$$K(P,Q) = \sum_{n=2}^{\infty} k_n \left(\frac{R_e^2}{r_P\, r_Q}\right)^{n+1} P_n(\cos\psi) \qquad (3.8)$$

where

$$k_n = (2n+1) \sum_{m=1}^{M} \frac{D_m^3 (2R_e - D_m)^3 K_{0m}}{(R_e - D_m)^2 (R_e^4 - (R_e - D_m)^4)} \left(1 - \frac{D_m}{R_e}\right)^{n+1}, \; n > 1 \qquad (3.9)$$

are the potential degree variances referring to the mean earth sphere.

The parameters were determined, in part, by fitting the covariance function model for the gravity disturbances to an empirical point anomaly covariance function. The latter was derived from a 1° mean anomaly covariance function (including corrections to the low-degree coefficients) of Tscherning and Rapp (1974). (The precise assumptions and the subsequent procedures that were applied to determine the parameters are not clearly presented in the reference above. Certainly, an approximation is involved here; namely, the covariance model for gravity disturbances was used in place of the model for gravity anomalies.)

The values, as given by Jordan, for the parameters of a 5-shell model are listed in Table I:

Table I: Parameters of Model (3.9)*

m	Co_m ($mgal^2$)	D_m(km)
1	595.4	16
2	812.3	93
3	216.1	391
4	166.4	1900
5	29.2	4780
total	1819.3	

* see equation (3.10)

Here, Co_m is the contribution to the variance of gravity anomalies on the sphere of radius R_e due to the shell at depth D_m. The anomaly variances are related to the potential variances by the law of propagation of covariances:

$$Co_m = \left(-\frac{\partial}{\partial r_P} - \frac{2}{r_P}\right)\left(-\frac{\partial}{\partial r_Q} - \frac{2}{r_Q}\right) K(P,Q)\Big|_{\psi=0} \tag{3.10}$$

III.3 The Model of Moritz (1977)

We now return to the models which are founded on the observed variation in the harmonic coefficients. Without loss in generality, the entire development of the various covariance functions in the remainder of this section is based on a model for the point anomaly covariance function, only because gravity anomalies are the quantities most readily observable. It is not necessary to proceed in this manner. For example, if the potential is considered to be the cardinal quantity of the gravity field, then one could design a model for its covariance function and from it derive the other covariances.

A model such as (3.3) should be a function only of the degree of the coefficient (see the remark following equation (3.2)). From the report by Rapp (1977), it is noted that the degree variances (computed from GEM 7 potential coefficients) start typically as (see also Table III)

$$\begin{array}{lll} c_2 = 7.5 \text{ mgal}^2 & c_3 = 33.8 \text{ mgal}^2 & c_4 = 19.6 \text{ mgal}^2 \\ c_5 = 21.3 \text{ mgal}^2 & c_6 = 18.9 \text{ mgal}^2 & c_7 = 19.5 \text{ mgal}^2 \end{array} \tag{3.11}$$

($c_0 = c_1 = 0$). Evidently, it is difficult to incorporate the degree variance c_2 into the model without unreasonably deforming it. This coefficient is therefore frequently omitted and the (modified) covariance function is modeled as an infinite sum beginning with n = 3. In fact, this scheme can be extended by accepting the first k empirically determined degree variances and modeling only

those with degree greater than k. The resulting covariance function is then of k^{th} order and (if k is not too small) is intended to reflect the local characteristics of the gravity field.

The model upon which Tscherning and Rapp (1974) have elaborated is

$$c_n = \frac{\alpha_2 (n-1)}{(n-2)(n+B)}, \quad n \geq 3 \tag{3.12}$$

where α_2 is a positive number and B is a nonnegative integer (the notation here is such that it is consistent with subsequent formulas). This model enjoys the property that a closed expression (using a recursion formula with respect to B) can be derived for the covariance function. The values of α_2 and B are determined in a least square adjustment of "observed" degree variances which are obtained through equation (2.41). The potential coefficients for (2.41) are deduced from satellite data and/or gravimetry (see also Rapp 1977).

The model above is asymptotic to $\frac{1}{n}$ (for large n, c_n behaves like $\frac{1}{n}$). Moritz (1976) calls the corresponding C (P,Q) a logarithmic covariance function. Indeed, when (3.12) is substituted into equation (2.36), we get ($s = \frac{R_e^2}{r_P r_Q}$)

$$C(P,Q) = \sum_{n=3}^{\infty} \frac{\alpha_2 (n-1)}{(n-2)(n+B)} s^{n+2} P_n(\cos\psi) \sim \sum_{n=3}^{\infty} \frac{1}{n} s^{n+2} P_n(\cos\psi) \tag{3.13}$$

The second sum can be evaluated from the generating function of Legendre polynomials:

$$\frac{1}{\sqrt{1-2st+s^2}} = \sum_{n=0}^{\infty} P_n(t)\, s^n, \quad |s| < 1 \tag{3.14}$$

(in which $t = \cos\psi$). It is easily verified that through an integration

$$\sum_{n=1}^{\infty} \frac{1}{n} P_n(t)\, s^{n+2} = s^2 \ln \left[\frac{\text{const.}}{\sqrt{1-2st+s^2} + 1 - st} \right] \tag{3.15}$$

The constant of integration is equal to 2, because for $t = 1$, the sum is $-s^2 \ln(1-s)$. It is thus evident, that the covariance function is logarithmic in nature. Moritz (1976) shows that for a planar approximation, the curvature parameter corresponding to (3.15) is very large (the lengthy derivation of this fact is not repeated here). This implies that also the horizontal gradient variance tends to be large for the model (3.12) ($G_{0H} \approx 3500\ E^2$ with the parameters $\alpha_2 = 425.28\ \text{mgal}^2$, $B = 24$).

A second model for the anomaly degree variances, as described by Moritz (1976) is given as

$$c_n = \frac{n-1}{n+A}\,, \quad n \geq 3 \tag{3.16}$$

where A is also a nonnegative integer (although the possibility of $A = -2$ can be considered). The covariance function represented by this model behaves like a reciprocal distance function. That is, c_n in (3.16) is asymptotic to 1, and the generating function of the Legendre polynomials shows that

$$\sum_{n=0}^{\infty} P_n(t)\, s^{n+2} = \frac{s^2}{\sqrt{1 - 2st + s^2}} \tag{3.17}$$

The latter resembles a reciprocal distance. In a planar approximation, the curvature parameter of the covariance function (3.17) (and consequently the horizontal gradient variance) is relatively small (Moritz 1976). Although the low gradient variance may be a desirable feature (see section II.7), the model above is unacceptable in view of the actual apparent decrease in the degree variances. Equation (3.16) implies that c_n tends to a nonzero constant as $n \to \infty$.

In his (1976) and (1977) reports, Moritz proposes and carries out the computation for a model of the anomaly degree variances which is constructed from a linear combination of the two models (3.12) and (3.16) above. The motivation behind this scheme rests on the possibility of manufacturing a curvature parameter corresponding to a low gradient variance, while at the same time retaining the favorable characteristics of the model (3.12), such as a realistic attenuation of the degree variances.

Therefore, let

$$C(P,Q) = \alpha_1 \sum_{n=3}^{\infty} \frac{n-1}{n+A}\, s_1^{\,n+2} P_n(\cos\psi) + \alpha_2 \sum_{n=3}^{\infty} \frac{(n-1)}{(n-2)(n+B)}\, s_2^{\,n+2} P_n(\cos\psi) \tag{3.18}$$

where α_1, α_2 are positive numbers, A, B are nonnegative integers, and

$$s_1 = \frac{R_1^{\,2}}{r_P r_Q}\,, \quad s_2 = \frac{R_2^{\,2}}{r_P r_Q} \tag{3.19}$$

R_1 and R_2 are radii of the type R (see section I). Each of the covariance components in (3.18) is a positive definite, isotropic, homogeneous covariance function of the gravity anomalies. It is immediately evident that the sum of these

two components has these same properties, and consequently, C (P, Q) as defined in (3.18) is indeed a covariance function of the gravity anomalies.

Taken individually, each component should converge for any two points P and Q on or exterior to the earth's surface. All theoretical developments of section II have been based on a spherical approximation, and therefore, it can be argued that it is necessary for R_1 and R_2 to be less than R_e (= 6371 km). When applying the subsequent formulas in practice, however, one may encounter difficulties if the points of evaluation are located within this mean earth sphere (but still above the earth's surface). Hence, it may be advantageous to have R_1, $R_2 < R_B$, where R_B is the radius of the Bjerhammar sphere (which is entirely enclosed within the earth).

The degree variance model which corresponds to the covariance function (3.18) is now determined. It is not merely the linear combination of the models (3.12) and (3.16). Suppose that we write

$$s_1 = \frac{R_C^2}{r_P r_Q} \frac{R_1^2}{R_C^2} \text{ and } s_2 = \frac{R_C^2}{r_P r_Q} \frac{R_2^2}{R_C^2} \tag{3.20}$$

Then equation (3.18) can be formulated as

$$C(P,Q) = \sum_{n=3}^{\infty}\left[\alpha_1 \frac{n-1}{n+A}\left(\frac{R_1^2}{R_C^2}\right)^{n+2} + \alpha_2 \frac{n-1}{(n-2)(n+B)}\left(\frac{R_2^2}{R_C^2}\right)^{n+2}\right]\left(\frac{R_C^2}{r_P r_Q}\right)^{n+2} P_n(\cos\psi) \tag{3.21}$$

or we may put this in the form

$$C(P,Q) = \sum_{n=3}^{\infty} c_n \left(\frac{R_C^2}{r_P r_Q}\right)^{n+2} P_n(\cos\psi) \tag{3.22}$$

In this way, the degree variances have been extracted from the covariance function of equation (3.18); they are

$$c_n = \alpha_1 \frac{n-1}{n+A}\left(\frac{R_1^2}{R_C^2}\right)^{n+2} + \alpha_2 \frac{n-1}{(n-2)(n+B)}\left(\frac{R_2^2}{R_C^2}\right)^{n+2}, \quad n \geq 3 \tag{3.23}$$

Exactly the same result is obtained by substituting (3.18) into (2.37). To some extent, R_1 and R_2 have lost their distinction as radii and are now regarded primarily as parameters of the model, subject to certain constraints (R_1, $R_2 < R_C$).

From (3.21), which is equivalent to (3.18), it is evident that $C(P, Q)$ does not depend on R_c. However, from equation (3.22) and with the assumption of no knowledge of the model for c_n, one concludes that R_c is the radius of the sphere to which the degree variances refer.

It is readily seen that for $\alpha_1 = 0$, the covariance function (3.21) degenerates into the covariance function as modeled (using equation (3.12)) by Tscherning and Rapp (1974). It should be noted that it is incorrect to substitute $\alpha_1 = 0$ in the two-component model (3.23) in order to obtain the degree variance model (3.12) - instead, the degree variances should always be derived from the covariance function that defines them. That is, c_n in (3.23) refers to a sphere of radius R_c (even if $\alpha_1 = 0$). However, with the model (3.12), c_n refers to a sphere of radius R.

The particular forms of the models (3.12) and (3.16) have been constructed in such a manner that the infinite sums can be reduced algebraically to closed expressions. Thereby, it becomes a routine matter to incorporate observation equations into the least squares adjustment for the point anomaly variance C_0, the mean anomaly variance $\bar{C}_0$, the gradient variance G_0, and the undulation variance L_0. The equation for L_0 was later deleted since the "observation" had been derived through equation (2.49) from the observed degree variances. Hence, L_0, thus computed, adds no new information to the model.

III.4 Closed Expressions For The Covariance Functions

Extensive computational formulas for the closed expressions of the gravity anomaly covariance function and the corresponding variance have been developed by Moritz (1977) for the model (3.23). Some of these formulas are reiterated here and others are derived, particulartly with respect to the gravity gradient. Several results are also borrowed from Tscherning and Rapp (1974).

To derive closed expressions of infinite series involving Legendre polynomials, we first define some elementary functions (series) for which closed expressions are known to exist. Let

$$F(s, \psi) = \sum_{n=0}^{\infty} s^{n+1} P_n(t) = \frac{s}{\sqrt{1 - 2st + s^2}} \qquad (3.24)$$

where $t = \cos\psi$; $|s| < 1$, $|t| \le 1$.

$$F_i(s, \psi) = \sum_{n=0}^{\infty} \frac{1}{n+i} s^{n+1} P_n(t) \text{ , for } i > 0 \qquad (3.25)$$

$$F_{-i}(s,\psi) = \sum_{n=i+1}^{\infty} \frac{1}{n-i}\, s^{n+1} P_n(t) \;, \quad \text{for } i \geq 0 \tag{3.26}$$

When $t = 1$ ($\psi = 0$), then $P_n(1) = 1$, for all n, and

$$F_i(s,0) = \sum_{n=0}^{\infty} \frac{1}{n+i}\, s^{n+1} = s^{1-i} \sum_{n=0}^{\infty} \int_0^s \zeta^{n+i-1}\, d\zeta = s^{1-i} \int_0^s \sum_{n=i-1}^{\infty} \zeta^n \, d\zeta =$$

$$= -s^{1-i} \ln(1-s) - \sum_{n=1-i}^{-1} \frac{1}{n+i}\, s^{n+1} \;, \quad i > 0 \tag{3.27}$$

similarly,

$$F_{-i}(s,0) = \sum_{n=i+1}^{\infty} \frac{1}{n-i}\, s^{n+1} = s^{1+i} \sum_{n=i+1}^{\infty} \int_0^s \zeta^{n-i-1}\, d\zeta = s^{1+i} \int_0^s \sum_{n=0}^{\infty} \zeta^n \, d\zeta$$

$$= -s^{1+i} \ln(1-s) \;, \quad i \geq 0 \tag{3.28}$$

The functions F_i, $i > 0$ and F_{-i}, $i \geq 0$ when $\psi \neq 0$ have been treated in Tscherning and Rapp (1974). The corresponding closed expressions are found essentially by integrating

$$\int \frac{s^{i-1}}{L}\, ds = \int \sum_{n=0}^{\infty} s^{n+i-1} P_n(t)\, ds = s^{i-1} F_i \;, \quad i > 0 \tag{3.29}$$

and

$$\int \frac{s^{-i-1}}{L}\, ds = \int \sum_{n=0}^{\infty} s^{n-i-1} P_n(t)\, ds = s^{-i-1} F_{-i} \;, \quad i \geq 0 \tag{3.30}$$

where $L = \sqrt{1 - 2st + s^2}$. Further letting $M = 1 - L - st$ and $N = 1 + L - st$, one can show that

$$\left.\begin{aligned} F(s,\psi) &= \frac{s}{L} \\ F_2(s,\psi) &= s\left[\tfrac{1}{2} M(3ts+1) + s^2 \left(P_2(t) \ln \frac{2}{N} + \tfrac{1}{4}(1-t^2)\right)\right] \end{aligned}\right\} \tag{3.31}$$

$$
\left.
\begin{aligned}
F_{-1}(s,\psi) &= s\left(M + ts\,\ell n \frac{2}{N}\right) \\
F_{i+1}(s,\psi) &= \frac{1}{s\cdot i}\left(L + (2i-1)\,t\,F_i(s,\psi) - \frac{1}{s}(i-1)\,F_{i-1}(s,\psi)\right),\ i > 1 \\
\text{with} \qquad & \\
F_1(s,\psi) &= \ell n\left(1 + \frac{2s}{1-s+L}\right) \\
F_2(s,\psi) &= \frac{1}{s}\left(L - 1 + t\,F_1(s,\psi)\right)
\end{aligned}
\right\} \quad \begin{array}{c}(3.31)\\ (\text{cont.})\end{array}
$$

The derivations of the formulas for the gradient covariance are given explicitly below; those for the gravity anomaly and undulation covariances follow along similar lines, but are not as complex. The details of these are given by Moritz (1977). Unfortunately, closed expressions for the mean anomaly covariance function can not be found due to the presence of products of Legendre polynomials in the smoothing factor β_n^2 of the degree variances.

III.4.1 The Gradient Covariance Function

It is assumed throughout all derivations that the anomaly degree variances refer to the mean earth sphere. Then $R_c = R_e$ in equation (2.68), which is repeated here:

$$
G(P,Q) = \sum_{n=2}^{\infty} c_n \frac{(n+2)^2}{R_e^2}\left(\frac{R_e^2}{r_P\, r_Q}\right)^{n+3} P_n(\cos\psi) \tag{3.32}
$$

Omitting the second-degree term and inserting the model (3.23) for c_n (with $\sigma_1 = \frac{R_1^2}{R_e^2}$, $\sigma_2 = \frac{R_2^2}{R_e^2}$) yields

$$
G(P,Q) = \sum_{n=3}^{\infty}\left[\alpha_1 \frac{n-1}{n+A}\sigma_1^{n+2} + \alpha_2 \frac{n-1}{(n-2)(n+B)}\sigma_2^{n+2}\right]\frac{(n+2)^2}{R_e^2}\left(\frac{R_e^2}{r_P\, r_Q}\right)^{n+3} P_n(\cos\psi) =
$$

$$= \frac{\alpha_1}{R_1^2} \sum_{n=3}^{\infty} \frac{(n-1)(n+2)^2}{n+A} s_1^{n+3} P_n(\cos\psi) +$$

$$+ \frac{\alpha_2}{R_2^2} \sum_{n=3}^{\infty} \frac{(n-1)(n+2)^2}{(n-2)(n+B)} s_2^{n+3} P_n(\cos\psi)$$

$$G(P,Q) = \alpha_1 G_1(P,Q) + \alpha_2 G_2(P,Q) \tag{3.33}$$

This equation and subsequent formulas based on it are slightly erroneous as presented in Moritz (1977, p. 16, equ. 3-24, 3-25). The difference is in the coefficients of the sums; Moritz gives them as α_i/R_e^2, $i = 1, 2$. Although most (except the final) numerical results of the least squares adjustment are obtained with the formulas exactly as given by Moritz, the correct results would hardly differ from these.

From equation (A-1) of this reference, we have

$$\frac{(n-1)(n+2)^2}{n+A} = (4-A)n + A(A-3) - \frac{(A+1)(A-2)^2}{n+A} + (n-1)n \tag{3.34}$$

$$\frac{(n-1)(n+2)^2}{(n-2)(n+B)} = n + 5 - B + \frac{16}{(B+2)(n-2)} + \frac{(B+1)(B-2)^2}{(B+2)(n+B)} \tag{3.35}$$

These can be readily derived through the use of long division and partial fractions. It will be convenient to introduce the following abbreviations:

$$\left.\begin{aligned} J_0 &= \sum_{n=3}^{\infty} s^{n+3} P_n(t), \quad J_1 = \sum_{n=3}^{\infty} n s^{n+3} P_n(t), \quad J_2 = \sum_{n=3}^{\infty} n^2 s^{n+3} P_n(t), \\ I_{-2} &= \sum_{n=3}^{\infty} \frac{1}{n-2} s^{n+3} P_n(t), \quad I_k = \sum_{n=3}^{\infty} \frac{1}{n+k} s^{n+3} P_n(t), \quad k > 0 \end{aligned}\right\} \tag{3.36}$$

Then from (3.24) - (3.26)

$$\left.\begin{aligned}
J_0 &= s^2 \sum_{n=3}^{\infty} s^{n+1} P_n(t) = s^2 \left(F - (s + s^2 t + s^3 P_2(t))\right) \\
I_{-2} &= s^2 F_{-2} \\
I_k &= s^2 \left(F_k - \left(\frac{s}{k} + \frac{s^2 t}{k+1} + \frac{s^3 P_2(t)}{k+2} \right) \right), \quad k > 0
\end{aligned}\right\} \quad (3.37)$$

To compute J_1, J_2, we require $\frac{\partial F}{\partial s}$, $\frac{\partial^2 F}{\partial s^2}$. Let

$$F = \frac{s}{L}, \quad \text{where } L = \sqrt{1 - 2st + s^2}$$

Now with $\frac{\partial L}{\partial s} = \frac{s-t}{L}$, the first derivative is

$$\frac{\partial F}{\partial s} = \frac{1 - st}{L^3} \quad (3.38)$$

and the second derivative simplifies to

$$\frac{\partial^2 F}{\partial s^2} = \frac{2tL^2 - 3s(1 - t^2)}{L^5} \quad (3.39)$$

Differentiating the sum which defines F yields

$$\frac{\partial F}{\partial s} = \sum_{n=0}^{\infty} (n+1)\, s^n P_n(t) =$$

$$= \sum_{n=3}^{\infty} n s^n P_n(t) + \sum_{n=3}^{\infty} s^n P_n(t) + 1 + 2st + 3s^2 P_2(t) =$$

$$= \frac{1}{s^3}(J_1 + J_0) + 1 + 2st + 3s^2 P_2(t) \quad (3.40)$$

Therefore,

$$J_1 = s^3 \frac{\partial F}{\partial s} - J_0 - s^3 (1 + 2st + 3s^2 P_2(t)) \quad (3.41)$$

Similarly, by differentiating again,

$$\frac{\partial^2 F}{\partial s^2} = \sum_{n=3}^{\infty} n^2 s^{n-1} P_n(t) + \sum_{n=3}^{\infty} n s^{n-1} P_n(t) + 2t + 6s P_2(t) =$$

$$= \frac{1}{s^4}(J_2 + J_1) + 2t + 6s P_2(t) \quad (3.42)$$

so that, finally

$$J_2 = s^4 \frac{\partial^2 F}{\partial s^2} - J_1 - s^4 (2t + 6s\, P_2(t)) \tag{3.43}$$

If the decomposed fractions (3.34) and (3.35) are substituted into the covariance function (3.33), then with the notation of (3.36), we arrive at the closed expressions for the vertical **gradient** covariance components (for A, B > 0)

$$\left.\begin{aligned} G_1(P,Q) &= \frac{1}{R_1^2}\Big[(A-3)(A J_0 - J_1) - (A+1)(A-2)^2 I_A + J_2 \Big]\Big|_{s_1} \\ G_2(P,Q) &= \frac{1}{R_2^2}\Big[J_1 + (5-B) J_0 + \frac{16}{B+2} I_{-2} + \frac{(B+1)(B-2)^2}{B+2} I_B \Big]\Big|_{s_2} \end{aligned}\right\} \tag{3.44}$$

where G_1 is evaluated at s_1 and G_2 is evaluated at s_2 .

The horizontal gradient variance on the sphere of radius R_e is derived in Moritz (1977), and with the correction mentioned above, it is given by

$$G_{0H} = \alpha_1 G_{10H} + \alpha_2 G_{20H}$$

where

$$\left.\begin{aligned} G_{10H} = \frac{\sigma_1^2}{2R_1^2}\Bigg[& \left[\frac{2\sigma_1^2}{(1-\sigma_1)^3} - 2\sigma_1^3\right] + (4-A)\left[\frac{\sigma_1^2}{(1-\sigma_1)^2} - \sigma_1^2 - 2\sigma_1^3\right] + \\ & + A(A-3)\frac{\sigma_1^4}{1-\sigma_1} + (A+1)(A-2)^2\left[\sigma_1^{1-A}\ln(1-\sigma_1) + \sum_{n=1-A}^{2}\frac{\sigma_1^{n+1}}{n+A}\right]\Bigg] \\ G_{20H} = \frac{\sigma_2^2}{2R_2^2}\Bigg[& \frac{\sigma_2^2}{(1-\sigma_2)^2} - \sigma_2^2 - 2\sigma_2^3 + (5-B)\frac{\sigma_2^4}{1-\sigma_2} + \\ & - \frac{16}{B+2}\sigma_2^3\ln(1-\sigma_2) - \frac{B+1}{B+2}(B-2)^2\left[\sigma_2^{1-B}\ln(1-\sigma_2) + \sum_{n=1-B}^{2}\frac{\sigma_2^{n+1}}{n+B}\right]\Bigg] \end{aligned}\right\} \tag{3.45}$$

Without the factor $\frac{1}{2}$, these equations would represent the vertical gradient variance. In the least squares adjustment (sec. III.6), the linearization of the problem requires the derivatives with respect to σ_1 and σ_2 ; they are (recalling that $\sigma_1 = R_1^2/R_e^2$, $\sigma_2 = R_2^2/R_e^2$)

$$\begin{aligned} \frac{dG_{10H}}{d\sigma_1} = \frac{1}{2R_e^2}\Bigg[& \frac{2\sigma_1^3(4-\sigma_1)}{(1-\sigma_1)^4} - 8\sigma_1^3 + (4-A)\left[\frac{\sigma_1^2(3-\sigma_1)}{(1-\sigma_1)^3} - 3\sigma_1^2 - 8\sigma_1^3\right] + \\ & + A(A-3)\frac{\sigma_1^4(5-4\sigma_1)}{(1-\sigma_1)^2} + (A+1)(A-2)^2\Bigg[(2-A)\sigma_1^{1-A}\ln(1-\sigma_1) + \\ & - \frac{\sigma_1^{2-A}}{1-\sigma_1} + \sum_{n=1-A}^{2}\frac{n+2}{n+A}\sigma_1^{n+1}\Bigg]\Bigg] \end{aligned} \tag{3.46}$$

$$\frac{d G_{20H}}{d\sigma_2} = \frac{1}{2R_e^2}\left[\frac{\sigma_2^2(3-\sigma_2)}{(1-\sigma_2)^3} - 3\sigma_2^2 - 8\sigma_2^3 + (5-B)\frac{\sigma_2^4(5-4\sigma_2)}{(1-\sigma_2)^2} + \right.$$

$$- \frac{64}{B+2}\sigma_2^3 \ln(1-\sigma_2) + \frac{16}{B+2}\frac{\sigma_2^4}{1-\sigma_2} - \frac{B+1}{B+2}(B-2)^2 \cdot \tag{3.46 (cont.)}$$

$$\left. \cdot \left[(2-B)\sigma_2^{1-B}\ln(1-\sigma_2) - \frac{\sigma_2^{2-B}}{1-\sigma_2} + \sum_{n=1-B}^{2}\frac{n+2}{n+B}\sigma_2^{n+1}\right]\right]$$

Also, $\frac{dG_{10H}}{d\sigma_2} = 0, \quad \frac{dG_{20H}}{d\sigma_1} = 0.$

III. 4. 2 Gravity Anomaly Covariance Function

The covariance function for the gravity anomalies is derived in an entirely analogous manner. Only the final results are quoted here. From equation (3.18)

$$\begin{aligned} C(P,Q) &= \alpha_1\sum_{n=3}^{\infty}\frac{n-1}{n+A}s_1^{\,n+2}P_n(\cos\psi) + \alpha_2\sum_{n=3}^{\infty}\frac{n-1}{(n-2)(n+B)}s_2^{\,n+2}P_n(\cos\psi) \\ &= \alpha_1 C_1(P,Q) + \alpha_2 C_2(P,Q) \end{aligned} \tag{3.47}$$

where $s_1 = \frac{R_1^2}{r_P r_Q}$, $s_2 = \frac{R_2^2}{r_P r_Q}$. Then Moritz (1977) gives (for $A, B > 0$)

$$\left.\begin{aligned} C_1(P,Q) &= s_1(F(s_1,\psi) - s_1 - s_1^2 t - s_1^3 P_2(t)) + \\ &\quad - (A+1)s_1\left(F_A(s_1,\psi) - \frac{s_1}{A} - \frac{s_1^2 t}{A+1} - \frac{s_1^3 P_2(t)}{A+2}\right) \\ C_2(P,Q) &= \frac{1}{B+2}s_2 F_{-2}(s_2,\psi) + \frac{B+1}{B+2}s_2\left(F_B(s_2,\psi) - \frac{s_2}{B} - \frac{s_2^2 t}{B+1} - \frac{s_2^3 P_2(t)}{B+2}\right) \end{aligned}\right\} \tag{3.48}$$

Here, F, F_A, F_B are defined in equations (3.24), (3.25), and $t = \cos\psi$.

The conditions $\psi = 0$ $(t = 1)$ and $r_P = r_Q = R_e$ result in the point anomaly variance on the sphere of radius R_e:

$$C_0 = \alpha_1 C_{10} + \alpha_2 C_{20}$$

$$\left.\begin{aligned} C_{10} &= \frac{\sigma_1^5}{1-\sigma_1} + (A+1)\sigma_1^{2-A}\,\ell n(1-\sigma_1) + (A+1)\sum_{n=1-A}^{2}\frac{\sigma_1^{n+2}}{n+A} \\ C_{20} &= -\frac{\sigma_2 + (B+1)\sigma_2^{2-B}}{B+2}\,\ell n(1-\sigma_2) - \frac{B+1}{B+2}\sum_{n=1-B}^{2}\frac{\sigma_2^{n+2}}{n+B} \end{aligned}\right\} \quad (3.49)$$

with

The derivatives of this variance with respect to σ_1 and σ_2 are

$$\left.\begin{aligned} \frac{dC_{10}}{d\sigma_1} &= \frac{\sigma_1^4(5-4\sigma_1)}{(1-\sigma_1)^2} + (A+1)(2-A)\sigma_1^{1-A}\,\ell n(1-\sigma_1) - (A+1)\frac{\sigma_1^{2-A}}{1-\sigma_1} + \\ &\quad + (A+1)\sum_{n=1-A}^{2}\frac{n+2}{n+A}\sigma_1^{n+1} \\ \frac{dC_{20}}{d\sigma_2} &= -\frac{4\sigma_2^3 + (B+1)(2-B)\sigma_2^{1-B}}{B+2}\,\ell n(1-\sigma_2) + \frac{\sigma_2^4 + (B+1)\sigma_2^{2-B}}{(B+2)(1-\sigma_2)} + \\ &\quad - \frac{B+1}{B+2}\sum_{n=1-B}^{2}\frac{n+2}{n+B}\sigma_2^{n+1} \end{aligned}\right\} \quad (3.50)$$

And $\dfrac{dC_{10}}{d\sigma_2} = 0, \quad \dfrac{dC_{20}}{d\sigma_1} = 0.$

III.4.3 Undulation Covariance Function

The covariance functions for the undulation and disturbing potential differ (in spherical approximation) only by the factor γ^2, where γ is an average value of gravity. Again, the potential covariance function is derived in Moritz (1977). Substituting the model (3.23) into equation (2.40) yields (with $R_c = R_e$)

$$k_n(R_e) = \frac{\alpha_1 R_e^2}{(n-1)(n+A)}\left(\frac{R_1^2}{R_e^2}\right)^{n+2} + \frac{\alpha_2 R_e^2}{(n-1)(n-2)(n+B)}\left(\frac{R_2^2}{R_e^2}\right)^{n+2} \quad (3.51)$$

The covariance function (2.28) is then

$$\begin{aligned} K(P,Q) &= \alpha_1\sum_{n=3}^{\infty}\frac{R_1^2}{(n-1)(n+A)}s_1^{n+1}P_n(\cos\psi) + \alpha_2\sum_{n=3}^{\infty}\frac{R_2^2}{(n-1)(n-2)(n+B)}s_2^{n+1}P_n(\cos\psi) \\ &= \alpha_1 K_1(P,Q) + \alpha_2 K_2(P,Q) \end{aligned} \quad (3.52)$$

For A, B > 0, Moritz (1977) derives

$$
\left.
\begin{aligned}
K_1(P,Q) &= \frac{R_1^2}{A+1}\left(F_{-1}(s_1,\psi) - s_1^3 P_2(t)\right) - \frac{R_1^2}{A+1}\left(F_A(s_1,\psi) - \frac{s_1}{A} - \frac{s_1^2 t}{A+1} - \frac{s_1^3 P_2(t)}{A+2}\right) \\
K_2(P,Q) &= \frac{R_2^2}{B+2} F_{-2}(s_2,\psi) - \frac{R_2^2}{B+1}\left(F_{-1}(s_2,\psi) - s_2^3 P_2(t)\right) + \\
&\quad + \frac{R_2^2}{(B+1)(B+2)}\left(F_B(s_2,\psi) - \frac{s_2}{B} - \frac{s_2^2 t}{B+1} - \frac{s_2^3 P_2(t)}{B+2}\right)
\end{aligned}
\right\} \quad (3.53)
$$

where $t = \cos\psi$, and F_{-1}, F_{-2}, F_A, F_B are defined by equations (3.31). The undulation covariance function is simply (see equ. (2.45) and (2.20))

$$
L(P,Q) = \frac{1}{\gamma_P \gamma_Q} K(P,Q) \quad (3.54)
$$

where $\gamma_P = \frac{kM}{r_P^2}$ and $\gamma_Q = \frac{kM}{r_Q^2}$. The undulation variance on the sphere of radius R_e is obtained by setting $\psi = 0$ and $r_P = r_Q = R_e$:

$$
L_0 = \frac{1}{\gamma_e^2}(\alpha_1 K_{10} + \alpha_2 K_{20}) \quad (3.55)
$$

where from (3.27) and (3.28), the variance components are

$$
\left.
\begin{aligned}
K_{10} &= \frac{-R_1^2}{A+1}\sigma_1^2\left(\ln(1-\sigma_1) + \sigma_1\right) + \frac{R_1^2}{A+1}\left(\sigma_1^{1-A}\ln(1-\sigma_1) + \sum_{n=1-A}^{2}\frac{\sigma_1^{n+1}}{n+A}\right) \\
K_{20} &= \frac{-R_2^2}{B+2}\sigma_2^3\ln(1-\sigma_2) + \frac{R_2^2}{B+1}\sigma_2^2\left(\ln(1-\sigma_2) + \sigma_2\right) + \\
&\quad - \frac{R_2^2}{(B+1)(B+2)}\left(\sigma_2^{1-B}\ln(1-\sigma_2) + \sum_{n=1-B}^{2}\frac{\sigma_2^{n+1}}{n+B}\right)
\end{aligned}
\right\} \quad (3.56)
$$

With $\sigma_1 = R_1^2/R_e^2$, $\sigma_2 = R_2^2/R_e^2$, these equations can be written more compactly as

$$
\left.
\begin{aligned}
K_{10} &= \frac{R_e^2\sigma_1}{A+1}\left(\left(\sigma_1^{1-A} - \sigma_1^2\right)\ln(1-\sigma_1) - \sigma_1^3 + \sum_{n=1-A}^{2}\frac{\sigma_1^{n+1}}{n+A}\right) \\
K_{20} &= \frac{R_e^2\sigma_2}{(B+1)(B+2)}\left[\left((B+2)\sigma_2^2 - (B+1)\sigma_2^3 - \sigma_2^{1-B}\right)\ln(1-\sigma_2) + (B+2)\sigma_2^3 - \sum_{n=1-B}^{2}\frac{\sigma_2^{n+1}}{n+B}\right]
\end{aligned}
\right\} \quad (3.57)
$$

The derivatives of this variance with respect to σ_1, and σ_2 are

$$
\frac{\partial L_0}{\partial\sigma_1} = \frac{\alpha_1}{\gamma_e^2}\frac{dK_{10}}{d\sigma_1} \quad , \quad \frac{\partial L_0}{\partial\sigma_2} = \frac{\alpha_2}{\gamma_e^2}\frac{dK_{20}}{d\sigma_2}
$$

where

$$\left.\begin{aligned}
\frac{dK_{10}}{d\sigma_1} &= \frac{R_e^2}{A+1}\left[\left((2-A)\,\sigma_1^{1-A} - 3\sigma_1^2\right)\ln(1-\sigma_1) - \frac{\sigma_1^{2-A} - \sigma_1^3}{1-\sigma_1} - 4\sigma_1^3 + \sum_{n=1-A}^{2} \frac{n+2}{n+A}\,\sigma_1^{\,n+1}\right] \\
\frac{dK_{20}}{d\sigma_2} &= \frac{R_e^2}{(B+1)(B+2)}\left[\left(3(B+2)\sigma_2^2 - 4(B+1)\sigma_2^3 - (2-B)\sigma_2^{1-B}\right)\ln(1-\sigma_2) + \right. \\
&\qquad \left. - \frac{(B+2)\sigma_2^3 - (B+1)\sigma_2^4 - \sigma_2^{2-B}}{1-\sigma_2} + 4(B+2)\sigma_2^3 - \sum_{n=1-B}^{2} \frac{n+2}{n+B}\,\sigma_2^{\,n+1}\right]
\end{aligned}\right\} \quad (3.58)$$

It is noted that none of the covariances above is defined for A or B equal to zero. By substituting $A = 0$ or $B = 0$ into the series expressions of the covariance functions, one can similarly derive closed formulas for these special cases (see Moritz 1977). However, they are of no importance in this study. (On the other hand, the variances as they stand above are in fact valid also for $A = 0$ or $B = 0$).

III.4.4 Mean Anomaly Covariance Function

As mentioned before, the mean anomaly covariance function (2.62) does not admit to such convenient summation. However, due to the smoothing factor $\beta_n^2 < O(n^{-2})$, this series expansion is expected to converge at a rate which will allow for truncation at a relatively low degree. Due primarily to the divisor $(1 - \cos\psi_0)^2$ in β_n^2, this rate obviously depends inversely on the size of the spherical cap. In fact, Tscherning and Rapp (1974) have shown through various tests that the mean anomaly covariance series may be terminated with sufficient accuracy at $N = 4\pi/\theta$ rad. $= 720°/\theta°$, where θ is the side of a mean anomaly block. Therefore, the approximation

$$\bar{C}(P,Q) \approx \sum_{n=3}^{720/\theta} \beta_n^2\, c_n\, s^{\,n+2}\, P_n(\cos\psi) \qquad (3.59)$$

is adopted here. With $R_c = R_e$ and model (3.23), this becomes

$$\bar{C}(P,Q) \approx \sum_{n=3}^{720/\theta} \beta_n^2 \left[\frac{\alpha_1(n-1)}{n+A}\left(\frac{R_1^2}{R_e^2}\right)^{n+2} + \frac{\alpha_2(n-1)}{(n-2)(n+B)}\left(\frac{R_2^2}{R_e^2}\right)^{n+2}\right]\left(\frac{R_e^2}{r_P r_Q}\right)^{n+2} P_n(\cos\psi) \qquad (3.60)$$

The corresponding variance on the sphere of radius R_e is computed by setting $\psi = 0$ and $r_P = r_Q = R_e$:

$$\overline{C}_0 \approx \sum_{n=3}^{720/\theta} \beta_n^2 \, c_n(R_e) \tag{3.61}$$

III.5 Second-Degree Terms

At this point, it is appropriate to list the contributions of the degree variance c_2 which have been omitted in all the model covariances. If c_2 refers to the mean earth sphere, then the second-degree term of the point anomaly covariance function is

$$c_2 \left(\frac{R_e^2}{r_P \, r_Q}\right)^4 P_2(\cos\psi) \tag{3.62}$$

For the other covariance functions, the second-degree terms are

$$\left.\begin{array}{lll} K(P,Q): & R_e^2 \, c_2 \left(\dfrac{R_e^2}{r_P \, r_Q}\right)^3 P_2(\cos\psi) & \\[2ex] L(P,Q): & \dfrac{R_e^2}{\gamma_P \gamma_Q} \, c_2 \left(\dfrac{R_e^2}{r_P \, r_Q}\right)^3 P_2(\cos\psi) \;, & \gamma_P = \dfrac{kM}{r_P^2} \;, \quad \gamma_Q = \dfrac{kM}{r_Q^2} \\[2ex] G(P,Q): & \dfrac{16}{R_e^2} \, c_2 \left(\dfrac{R_e^2}{r_P \, r_Q}\right)^5 P_2(\cos\psi) & \\[2ex] \overline{C}(P,Q): & \beta_2^2 \, c_2 \left(\dfrac{R_e^2}{r_P \, r_Q}\right)^4 P_2(\cos\psi) & \end{array}\right\} \tag{3.63}$$

III.6 The Least Squares Adjustment

The procedure to be followed is now apparent. For each of the observations, namely the anomaly coefficients c_n, $n = 3, 4, \ldots$ and the assorted variances, there exists a corresponding observation equation containing the parameters of the model. The parameters (unknowns) are α_1, α_2, A, B, σ_1, σ_2, but there are many more observations; an application of a least squares adjustment thereby suggests itself.

The observation equations (3.23), (3.61), (3.49), (3.55), and (3.45) are not linear with respect to most of the parameters, nor are the latter three equations continuous in the integer variables A and B. The problem of nonlinearity is circumvented, in principle, by linearizing the observation

equations and iterating the least squares solution until it is sufficiently close to the nonlinear solution. The parameters A and B cannot be adjusted analytically since it is impossible to differentiate the variances with respect to these parameters; therefore, one must simply assign various values to A, B and investigate the resulting behavior of the solution. The parameters of the adjustment are hence reduced to α_1, α_2, σ_1, σ_2.

The linearization is effected by expanding the system of observation equations in a Taylor series about some expansion point and truncating the series at the first-order terms. The coefficient of the first-order terms is the matrix of partial derivatives with respect to the parameters, and it is evaluated at the expansion point. The derivatives of the variances C_0, G_{0H}, L_0 have already been obtained above. They are

$$\frac{\partial(C_0, G_{0H}, L_0)}{\partial(\alpha_1, \alpha_2, \sigma_1, \sigma_2)} = \begin{pmatrix} C_{10} & C_{20} & \alpha_1 \frac{dC_{10}}{d\sigma_1} & \alpha_2 \frac{dC_{20}}{d\sigma_2} \\ G_{10H} & G_{20H} & \alpha_1 \frac{dG_{10H}}{d\sigma_1} & \alpha_2 \frac{dG_{20H}}{d\sigma_2} \\ \frac{1}{\gamma_e^2} K_{10} & \frac{1}{\gamma_e^2} K_{20} & \frac{\alpha_1}{\gamma_e^2} \frac{dK_{10}}{d\sigma_1} & \frac{\alpha_2}{\gamma_e^2} \frac{dK_{20}}{d\sigma_2} \end{pmatrix} \qquad (3.64)$$

Recalling the model for c_n (3.23), the derivatives with respect to the parameters are

$$\left.\begin{aligned} \frac{\partial c_n}{\partial \alpha_1} &= \frac{n-1}{n+A}\, \sigma_1^{\,n+2} \quad ; & \frac{\partial c_n}{\partial \alpha_2} &= \frac{n-1}{(n-2)(n+B)}\, \sigma_2^{\,n+2} \\ \frac{\partial c_n}{\partial \sigma_1} &= \alpha_1 \frac{(n-1)(n+2)}{n+A}\, \sigma_1^{\,n+1} \quad ; & \frac{\partial c_n}{\partial \sigma_2} &= \frac{(n-1)(n+2)}{(n-2)(n+B)}\, \sigma_2^{\,n+1} \end{aligned} \quad n = 3, 4, \ldots \right\} \qquad (3.65)$$

Since the smoothing factor β_n^2 in the mean anomaly variance is independent of the parameters, equation (3.61) yields

$$\frac{\partial \bar{C}_0}{\partial x_i} = \sum_{n=3}^{720/\theta} \frac{\partial c_n}{\partial x_i}\, \beta_n^2 \quad , \qquad i = 1, \ldots, 4 \qquad (3.66)$$

where the x_i, $i = 1$, ..., 4 represent the four unknowns α_1, α_2, σ_1, σ_2, respectively.

Let Γ_s be the q-vector of "linearized observations", let D denote the q x 4 coefficient matrix of derivatives, and let Γ_0 be the vector of values implied by the (nonlinear) model for a particular set of parameters X_0. Then the linearized system of equations is

$$\Gamma_a(X_a) = \Gamma_O + D(X_a - X_O) \qquad (3.67)$$

The least squares solution then determines the parameters X_a for which the sum of squares of the differences (i.e. residuals) between the actual observations Γ_b and the values Γ_O that are implied by the model is a minimum. The adjustment is actually carried out with a provision for weighting both the observations and the parameters.

With reference to Uotila (1967), the adjusted parameters of the i-th iteration are

$$X_{a\,i} = X_{a\,i-1} + X_i\,, \qquad i > 0 \qquad (3.68)$$

with

$$X_i = -(D_{i-1}^T P_b D_{i-1} + P_x)^{-1} (D_{i-1}^T P_b \Gamma_{i-1} + P_x H_{i-1}) \qquad (3.69)$$

P_b, P_x are the weight matrices for the observations and parameters, respectively. These matrices are diagonal since it is assumed that no correlation exists among the anomaly degree variances; also there is no correlation between the variances because they are determined independently. D_{i-1} is the coefficient matrix evaluated at $X_{a_{i-1}}$; $\Gamma_{i-1} = \Gamma_{O_{i-1}} - \Gamma_b$, where $\Gamma_{O_{i-1}}$ is the vector of "computed observations" that are obtained from the parameters $X_{a_{i-1}}$ in the nonlinear model; and $H_{i-1} = X_{a_{i-1}} - X_O$, where $X_{a\,O} \equiv X_O$ (the original expansion point). Also Γ_{O_i} is the vector of adjusted observations.

IV. Numerical Results

IV.1 The Variation of $c_n(R_c)$ as R_c Changes

In the previous section, it was assumed that the anomaly degree variances refer to the mean earth sphere ($R_c = R_e$). This assumption simplifies many of the expressions for the variances which are generally computed as referring to the geoid ($\sim$ mean earth sphere); then $s = R_c^2/R_e^2 = 1$. The observed degree variances, however, are computed from equation (2.41) using potential coefficients. Hence, they refer to the Bjerhammar sphere, since the R in the disturbing potential (equation (2.22)) is usually identified with the Bjerhammar sphere radius R_B. It is appropriate then to investigate the variation of $c_n(R_c)$, computed by equation (2.39), as R_c changes. Comparing (2.39) and (2.41), we have

$$c_n(R_c) = c_n'\left(\frac{R^2}{R_c^2}\right)^{n+2} \tag{4.1}$$

where c_n' does not depend on R_c. Therefore,

$$d c_n(R_c) = -(n+2)\, c_n(R_c)\, \frac{d(R_c^2)}{R_c^2} \tag{4.2}$$

or approximately

$$\Delta c_n = -(n+2)\, c_n\, \frac{\Delta R_c^2}{R_c^2} \tag{4.3}$$

We take $R_c = R_e = 6371$ km and a relatively large difference

$$\Delta R_c^2 = R_e^2 - b^2 \approx 1.8 \times 10^5 \text{ km}^2, \tag{4.4}$$

where $b = a(1-f)$ and a, f are the parameters of a mean earth ellipsoid ($a = 6378.140$ km, $1/f = 298.257 \rightarrow b = 6356.755288$ km). Table II illustrates the corresponding decrease in the degree variances. The values for c_n are given by Rapp (1977, p. 40) and actually refer to the Bjerhammar sphere, but this is immaterial for these computations which serve only as an example.

Table II : Variation of c_n (as given by (4.3)) with change in R_c^2 (all values in mgal^2)

n	c_n	Δc_n	$c_n + \Delta c_n$
3	33.2	-.745	32.46
7	16.7	-.674	16.03
13	5.6	-.377	5.22
20	2.4	-.237	2.16
30	2.8	-.402	2.40
40	3.9	-.735	3.17
50	4.5	-1.050	3.45

The change Δc_n is directly proportional to ΔR_c^2. To determine ΔR_c^2 for which Δc_n might be considered negligible, one requires that

$$\Delta R_c^2 < \frac{t}{q} (R_e^2 - b^2) \tag{4.5}$$

where t is the tolerance on Δc_n, and q is the maximum value of $|\Delta c_n|$ for $\Delta R_c^2 = R_e^2 - b^2$. For $n \leq 40$, $q = .745$ and specifying $t = .05$, it is seen that if

$$\Delta R_c^2 < \frac{.05}{.745} (R_e^2 - b^2) \approx 12100 \text{ km}^2$$

then the c_n values for such a ΔR_c^2 are not affected in the first decimal place (since the correction Δc_n is then $< .05$). Because

$$\Delta R_c^2 = (R_e^2 - b^2) = (R_e - b)(R_e + b) = \Delta R_c (R_e + b),$$

the values $\Delta R_c^2 < 12100 \text{ km}^2$ correspond to $\Delta R_c < .95$ km.

Therefore, if the observed degree variances refer to the Bjerhammar sphere whose radius may deviate from R_e by more than 10 km, then it is necessary to implement equation (4.1) with $R_c = R_e$ to obtain degree variances which refer to the mean earth sphere. From Table II and the subsequent discussion, it is evident that if R_B is close to the semiminor axis of the mean earth ellipsoid, then for low degree, the error in using $R = b$ instead of $R = R_B$ does not affect the first decimal of the values of c_n. It is assumed then that a sphere with radius b is close to a sphere embedded entirely within the earth.

IV.2 The Observed Data

IV.2.1 Anomaly Degree Variances

One set of observed degree variances, c_n', has been computed by Rapp (1977 p.40). The potential coefficients upon which they are based were de-

Table III: Anomaly Degree Variances from Rapp(1977)

(all values in $mgal^2$)

n	c_n'	c_n *	stand. dev.	n	c_n'	c_n *	stand. dev.
3	33.2	32.47	.8	28	2.9	2.54	.8
4	14.9	14.51	.8	29	3.6	3.13	.8
5	11.1	10.76	.8	30	2.8	2.43	.8
6	21.5	20.74	.8	31	2.2	1.90	.8
7	16.7	16.04	.8	32	3.3	2.83	.8
8	7.0	6.69	.8	33	2.7	2.31	.8
9	14.7	13.99	.8	34	3.6	3.06	.8
10	8.7	8.24	.8	35	3.2	2.71	.8
11	8.3	7.83	.8	36	3.9	3.29	.8
12	3.1	2.91	.8	37	3.0	2.52	.8
13	5.6	5.24	.8	38	3.8	3.18	.8
14	3.6	3.35	.8	39	2.8	2.33	.8
15	5.2	4.82	.8	40	3.9	3.23	.8
16	5.1	4.71	.8	41	3.5	2.89	.8
17	4.4	4.04	.8	42	3.7	3.04	.8
18	3.7	3.38	.8	43	4.1	3.35	.8
19	3.4	3.09	.8	44	4.1	3.34	.8
20	2.4	2.17	.8	45	3.4	2.75	.8
21	3.0	2.71	.8	46	3.2	2.58	.8
22	3.4	3.05	.8	47	3.8	3.05	.8
23	2.7	2.41	.8	48	3.8	3.04	.8
24	2.4	2.14	.8	49	3.7	2.94	.8
25	2.7	2.39	.8	50	4.5	3.57	.8
26	2.6	2.29	.8	51	4.4	3.47	.8
27	2.2	1.93	.8	52	4.5	3.53	.8

*modified according to equ. (4.1) with $R_o = R_e$, $R = b = 6356.755288$ km

duced from a global set of 5° mean anomalies (which were obtained from a set of 1° anomalies). If one multiplies equation (2.60) on both sides by

$$\overline{P}_{ij}(\cos\overline{\theta})\begin{Bmatrix}\cos j\overline{\lambda}\\ \sin j\overline{\lambda}\end{Bmatrix} \tag{4.6}$$

and integrates over the sphere, then with the aid of the orthogonality properties of Legendre functions, one can show (Heiskanen and Moritz 1967, p. 31) that the potential coefficients are given by (let $R = R_\theta$ in equation (2.60))

$$\begin{Bmatrix} \overline{C}_{ij} \\ \overline{S}_{ij} \end{Bmatrix} = \frac{R_c^{\,i+2}}{4\pi\gamma_0(i-1)\beta_i R_\theta^{\,i+2}} \iint_\sigma \overline{\Delta g}(\overline{\theta},\overline{\lambda})\, \overline{P}_{ij}(\cos\overline{\theta}) \begin{Bmatrix} \cos j\overline{\lambda} \\ \sin j\overline{\lambda} \end{Bmatrix} d\sigma \qquad (4.7)$$

where β_i is the smoothing factor for 5° anomalies. Since the mean anomalies are in the form of discrete values, the integration is replaced by a summation, in which case, $\overline{\Delta g}$ is evaluated only at the center points of disjoint blocks (see Rapp 1977). Substituting these coefficients into (2.41) then yields the degree variances c_n' as listed above in Table III. These values are modified using equation (4.1) with $R_c = R_e = 6371$ km and $R = b = 6356.755288$ km, see p. 39, in order that they refer to the mean earth sphere. The selection of the standard deviation is discussed below.

A second set of anomaly degree variances can be derived from the potential coefficients of the GEM 9 satellite solution (Lerch et al. 1977). Applying the same procedure as above yields the results listed in Table IV.

Table IV: Anomaly Degree Variance from GEM 9 Potential Coefficients (all values in $mgal^2$)

n	c_n'	c_n *	stand. dev.	n	c_n'	c_n *	stand. dev.
3	33.66	32.91	.8	12	3.67	3.44	.8
4	19.63	19.11	.8	13	6.59	6.16	.8
5	20.87	20.22	.8	14	4.04	3.76	.8
6	19.05	18.38	.8	15	3.30	3.06	.8
7	19.45	18.68	.8	16	2.34	2.16	.8
8	11.73	11.21	.8	17	2.05	1.88	.8
9	11.50	10.95	.8	18	3.32	3.03	.8
10	10.07	9.54	.8	19	2.99	2.72	.8
11	6.77	6.39	.8	20	2.30	2.08	.8

* modified according to equation (4.1) with $R_c = R_e$, $R = b = 6356.755288$ km

It should be remarked that the modification of the anomaly degree variances through equation (4.1) is based on the presupposition that the potential coefficients refer to a sphere of radius $R = b$. This assumption may be questionable, particularly with respect to the coefficients for the degree variances of Table III. These coefficients were computed using equation (4.7) with $R_e^2/R_\theta^2 \approx 1$ (see Rapp 1977).

IV.2.2 Point Anomaly Variance

The point anomaly variance has been estimated by Tscherning and Rapp (1974) to be ~ 1800 mgal^2. This value is based on over 2.25 million free-air anomalies which were partitioned according to the ranges of elevation in which they were determined. A weighted average of the individual variances of the anomaly subdivisions then provided the total variance. The effect of c_2 on C_0 is c_2 itself (≈ 7.5 mgal^2, see equation (3.62)), if c_2 refers to a sphere of radius R_e. It is neglected in the input value to the adjustment, since C_0 does not act as a constraint in the strictest sense (a minimum standard deviation of 25 mgal^2 was attached to C_0). Therefore, the exact input value is not too critical. The final adjusted value of C_0 is one that is partly implied by the other data and the model. The same is true of the undulation variance after the second-degree contribution has been removed.

IV.2.3 Undulation Variance

The value of L_0 is obtained simply by summing the first few undulation degree variances. The sum in equation (2.49) converges rapidly, and with the observed anomaly degree variances (e.g. computed from GEM 9 potential coefficients (Lerch et al. 1977)), it results in an approximate, rounded value of 900 m^2. The effect of c_2 on L_0 is sizable, and in this case, it must be subtracted to accommodate the model. This effect (equation (3.63)) is approximately

$$\frac{R^2}{\gamma^2} c_2 = \frac{R^6}{(kM)^2} c_2 = 314 \text{ m}^2 \tag{4.8}$$

with $R = b$, $kM = 398601$ km^3/sec^2, and $c_2 = 7.56$ mgal^2. By the manner in which the input value L_0 is computed, one cannot expect it to add new information to the adjustment. The observation for L_0 was therefore deleted in the final analysis (Table IX); but it was included in the solutions of Tables VI and VIII. Its removal had no significant influence on the numerical results.

IV.2.4 Mean Anomaly Variances

The mean anomaly variances for 1° and 5° blocks are also derived by Rapp (1977). The covariance function (and hence the variance, when $\psi = 0$) for the 1° mean anomalies is estimated there similarly as the point anomaly variance was deduced by Tscherning and Rapp (1974); namely, by forming weighted averages of given mean anomaly data (Land/Ocean-value, p. 7, Rapp 1977). 5° mean anomalies were then obtained through a least squares prediction using this 1° covariance function. The square of the resulting root mean square of 5° anomalies (set 1, p. 14, Rapp 1977) is adopted here as the

observation.

Again, because the summation of the model begins with n = 3, it is necessary to subtract the contribution of c_2 (see equation (3.63)):

$$\text{For } 1^\circ \text{ blocks: } \psi_0 = \sqrt{(1^\circ)^2/\pi} = 0^\circ.564 \rightarrow \beta_2^2 c_2 s^4 = 7.43 \text{ mgal}^2$$
$$\text{For } 5^\circ \text{ blocks: } \psi_0 = \sqrt{(5^\circ)^2/\pi} = 2^\circ.821 \rightarrow \beta_2^2 c_2 s^4 = 7.41 \text{ mgal}^2 \qquad (4.9)$$

where $c_2 = 7.5$ mgal^2 is assumed to refer to a sphere of radius R = b, and $s = R^2/R_e^2$ (for β_2, see equation (2.58)).

Table V below displays the observational values of the variances discussed above (for the choice of the standard deviation, see below).

Table V: Variances

observation	standard deviation
$C_0 = 1800$ mgal^2	200 mgal^2
$L_0 = 900 - 314 = 586$ m^2	50 m^2
$\bar{C}_0\|_{1^\circ} = 862.5 - 7.43 = 855.1$ mgal^2	10 mgal^2
$\bar{C}_0\|_{5^\circ} = 259.2 - 7.41 = 251.8$ mgal^2	10 mgal^2

As mentioned on page 16, one cannot expect the adjustment process to determine a value for G_{0H} based only on low-degree information such as the first 20 degree variances. Therefore, any input value G_0 must act as a constraint (i.e. with a weight relatively larger than for the other variances). By assigning a small weight to G_{0H}, the iterated least squares solution usually oscillated wildly and diverged. Even with the constraining standard deviations that were finally chosen, a slight divergence of the solution is detectable.

From the discussion in section II.7 on the observed variability of the horizontal gradient variance, it was decided to select two global values for G_{0H}: 200E^2 and 3500E^2. Each value implies a different set of model parameters. The initial computations were oriented towards determining a model from the degree variance data of Table III, with a low gradient variance (as opposed to the model of Tscherning and Rapp (1974) which yields a large gradient variance). Therefore, only the value $G_{0H} = 200$ E^2 was applied to the data of Tables III and V.

In view of equation (3.63), the influence of c_2 on G_{0H} is negligible. With $c_2 \approx 7.5$ mgal^2, $r_P = r_Q = R_e$, $\psi = 0$,

$$c_2 \frac{16}{R_0^2} \left(\frac{R_0^2}{r_P \, r_Q} \right)^5 P_2(\cos\psi) \; < \; 3 \times 10^{-4} \; E^2 \qquad (4.10)$$

IV.3 The Weights

The least squares adjustment is to be regarded, in this case, as a procedure which results in the best fit of the model to the given data. In this sense, the standard deviation which is assigned to an observation is not taken to represent a measure of accuracy, but rather, only as an indication of the relative weight or significance that the observation should carry. With this in mind, one can arbitrarily vary the weights to procure the best possible fit of the model. The original standard deviations are listed in Tables III, IV, and V for the input data. A standard deviation of .8 $(\text{mgal})^2$ (also used by Tscherning and Rapp (1974)) was assigned to every observed degree variance and retained throughout all computations. Some experiments were conducted to determine whether a change in the weights at the upper or lower end of the sequence of degree variances would improve the adjustment. Such variation in the weights usually did not enhance the fit of the degree variances, while often being detrimental also to the model variances. The standard deviations of the variances were initially selected arbitrarily. Later it became necessary to tighten the control on these quantities in order to improve the fit to the data (cf. Tables VI, VIII, IX).

The weights of the parameters α_1, α_2 were set practically to zero (10^{-12}), since these unknowns should be established entirely by the observed data. This is equally true for the parameters σ_1 and σ_2, but these were restrained by a standard deviation of 0.01 to prevent a possible underflow or overflow in the machine computations (e.g. the 1° anomaly variance contains the factor $(\sigma_1)^{722}$).

IV.4 Data Set of Table III, Two-Component Model

A least squares adjustment was applied to the observations that are collected in Tables III and V, and to $G_{0H} = 200 \; (\pm 50) \; E^2$, for various and sundry values of A and B. An investigation of the results revealed that the value $A \approx 100$ yields the most favorable solutions. Most multiples of 10 below, and some above A = 100 were also tested; in each instance, the solution was deemed unacceptable as it generally produced larger residuals than in the case with A = 100. A few selected results are presented in Table VI. In this table, RMS denoted the root mean square value, defined in principle by the square root of the simple average of squared quantities. In this context, RMS $(\bar{C}_0)$ is the RMS value of the residuals of both mean anomaly variances; while RMS (c_n) stands for the RMS value of residuals of the anomaly degree variances. The adjusted variances, as they are listed in Table VI, do not

Table VI: Least Squares Solutions for the Model (3.23) with the Data of Tables III and V.

Sln.		stand. dev. if diff. from Table V			Model Parameters		Variances implied by the Model						
		(C_0)	$(\bar{C}_{\sigma 1°})$	$(\bar{C}_{\sigma 5°})$ (mgal)^2	α_1, α_2 (mgal)^2	σ_1, σ_2	C_0 (mgal)^2	G_{OH} E^2	L_0 m^2	$\bar{C}_d(1°)$ (mgal)^2	$\bar{C}_d(5°)$ (mgal)^2	RMS$(\bar{C}_0)$ (mgal)^2	RMS(c_a) (mgal)^2
(a)	A=100 B= 10				16.1878 319.9062	.9954022 .9158591	2069.8	335.1	551.5	826.0	216.4	32.4	2.22
(b)	A=110 B= 10				17.2054 318.5533	.9953171 .9166125	2075.1	330.6	551.3	827.5	215.9	32.1	2.21
(c)	A= 90 B= 10				15.1721 321.4978	.9954928 .9149711	2063.4	339.9	551.7	824.3	217.1	32.8	2.24
(d)	A=100 B= 15	75			15.9876 417.6274	.9952409 .9198024	1970.0	296.6	541.8	816.0	223.0	34.3	2.24
(e)	A=100 B= 10	75			16.3352 307.7572	.9951682 .9211232	1964.8	288.7	549.0	818.2	219.4	34.7	2.24
(f)	A=100 B= 5	75	3	3	17.3048 180.5496	.9949134 .9430983	1957.0	259.7	583.2	852.4	233.9	12.8	2.46
(g)	A=100 B= 20	75	3	3	16.3771 505.7053	.9952237 .9286496	2026.0	300.3	549.6	851.4	240.3	8.6	2.45
(h)	A=100 B= 10	75	3	3	17.1271 302.8482	.9949795 .9302030	1972.3	268.0	572.8	852.2	235.9	11.4	2.41
(i)	A=100 B= 10	75	6	3	16.9024 303.1426	.9949862 .9303730	1952.7	265.6	573.6	844.4	235.4	13.9	2.40
(j)	A= 0 B= 0				5.8514 71.0883	.9967419 .9104711	1833.3	414.1	557.2	775.8	241.7	56.5	2.77
(k)	A= 0 B= 0				9.907 37.770	.993963 .999281	1801.2	199.7	591.7	1085.3	361.9	180.4	5.74

include the value of the second-degree term of their series expansions.

The following criteria were considered in the selection of the "best" results: the adjusted values of C_0 and L_0 should be close to the observed values; G_{0H} should have a value approximately equal to 200 E^2; the values of RMS (c_n) and RMS ($\overline{C_0}$) should be minimal. It is also desirable to obtain values of σ_1 and σ_2 such that $R_1 < b$ and $R_2 < b$. For $R = b$,

$$\hat{\sigma} = \frac{R^2}{R_e^2} = .9955333 \tag{4.11}$$

Finally, the low-degree c_n's of the model should not be unreasonable distorted for the sake of accommodating the higher-degree values.

With these standards of selection, solution (i) of Table VI is judged to be most favorable for this data set. The results of (a), (b), and (c) show that $A \approx 100$, while solutions (d), (e), (f), (g), and (h) suggest that B should not exceed 10. For the solution (f) $(B = 5)$, the value of c_3 as obtained from the adjusted model is $c_3 = 34.0$ mgal^2; whereas, for (h) $(B = 10)$, $c_3 = 32.8$mgal^2. In view of the observed value of c_3 in Table III, it can be argued that $5 \le B \le 10$. It is noted that σ_1, $\sigma_2 < \hat{\sigma}$ (see (4.11)) in cases (a) - (i); hence, the covariance functions defined by these models converge for any two points on the earth, provided that b is the radius of a sphere contained within the earth.

The solutions of Table VI are based on degree variances as modified by equation (4.1) to refer to the mean earth sphere. If the degree variances c_n' of Table III had been treated as already referring to this sphere, then the results of Table VI would not be altered substantially. This data set primarily served in the general investigation of the two-component model (3.23). The final model parameters(sec. IV. 6) are determined from the anomaly degree variances implied by the GEM 9 potential coefficients.

The adjusted degree variances of model (3.23), referring to the sphere of radius R_e, and for solution (i) in Table VI are given below in Table VII.

It is observed in Table VI that the adjustment process seems to be able to determine a value for the horizontal gradient variance G_{0H}, apparently through the other variances. However, this is the case only when more weight is placed on the "observed" G_{0H} than on the other variances (particularly C_0). The adjusted value of C_0 was improved by increasing its weight, thereby necessitating even tighter control on G_{0H}. With the observed value of G_{0H} used here, a large value of the parameter A was required, in part, to push the "adjusted" G_{0H} down to ~200 E^2. (With the observed value of 3500 E^2, on the other hand, a large value was assigned to A in order to decrease C_0 to ~1800 mgal^2; see also Table IX.) The solution (j) of Table VI corresponds to the

parameters A, B as adopted by Moritz (1977). With respect to A, it reflects a limiting value of G_{OH} that is "implied" by the input (with an observed $G_{OH} = 200\ E^2$). That is, as A is increased, G_{OH} decreases. The last entries in Table VI are obtained with the model parameters that were determined by Moritz.

Table VII: Adjusted Anomaly Degree Variances (sln. (i), Table VI)

(all values in $mgal^2$)

n	c_n	residual	n	c_n	residual
3	32.83	.37	28	4.02	1.48
4	21.54	7.03	29	4.00	.87
5	16.88	6.12	30	3.99	1.56
6	14.06	-6.68	31	3.99	2.09
7	12.08	-3.96	32	3.99	1.15
8	10.59	3.90	33	3.99	1.68
9	9.42	-4.58	34	4.00	.94
10	8.47	.23	35	4.02	1.30
11	7.70	- .13	36	4.03	.74
12	7.07	4.15	37	4.05	1.53
13	6.54	1.30	38	4.07	.89
14	6.09	2.74	39	4.09	1.76
15	5.72	.90	40	4.11	.88
16	5.40	.70	41	4.14	1.25
17	5.14	1.10	42	4.16	1.12
18	4.92	1.54	43	4.18	.84
19	4.73	1.64	44	4.21	.88
20	4.58	2.40	45	4.24	1.49
21	4.45	1.74	46	4.27	1.69
22	4.34	1.29	47	4.29	1.24
23	4.25	1.84	48	4.32	1.28
24	4.18	2.04	49	4.35	1.40
25	4.12	1.73	50	4.37	.81
26	4.08	1.78	51	4.40	.93
27	4.04	2.11	52	4.42	.89

Some additional remarks concerning this model and the determination of its parameters are appropriate. With observation equations only for the degree variances (and mean anomaly variances), it would be difficult in the adjustment to separate the corresponding parameters of the two components of the model. But observation equations are included for C_O, G_O, L_O, which are essentially of a different character, thereby allowing the parameters to be separated, although not without any correlation. The correlation matrix for the parameters α_1, α_2, σ_1, σ_2 of solution (i) in Table VI is

$$
\begin{array}{c|cccc}
 & \alpha_1 & \alpha_2 & \sigma_1 & \sigma_2 \\
\hline
\alpha_1 & & & & \\
\alpha_2 & & & & \\
\sigma_1 & & & & \\
\sigma_2 & & & &
\end{array}
\quad
\begin{pmatrix}
1.00 & .55 & -.96 & -.74 \\
.55 & 1.00 & -.55 & -.92 \\
-.96 & -.55 & 1.00 & .70 \\
-.74 & -.92 & .70 & 1.00
\end{pmatrix}
$$

By inspecting the adjusted degree variances of Table VII for the two-component model, it is observed that their values increase for $32 \leq n \leq 52$, when in fact, c_n should approach zero as $n \to \infty$. The graph below clarifies the situation by showing the behavior of $c_n = c_{1\,n} + c_{2\,n}$ up to some large n, where

$$
c_{1\,n} = \alpha_1 \frac{n-1}{n+A} \sigma_1^{\,n+2} \quad , \quad c_{2\,n} = \alpha_2 \frac{n-1}{(n-2)(n+B)} \sigma_2^{\,n+2} \qquad (4.12)
$$

This verifies that the modeled $c_n \to 0$ as $n \to \infty$.

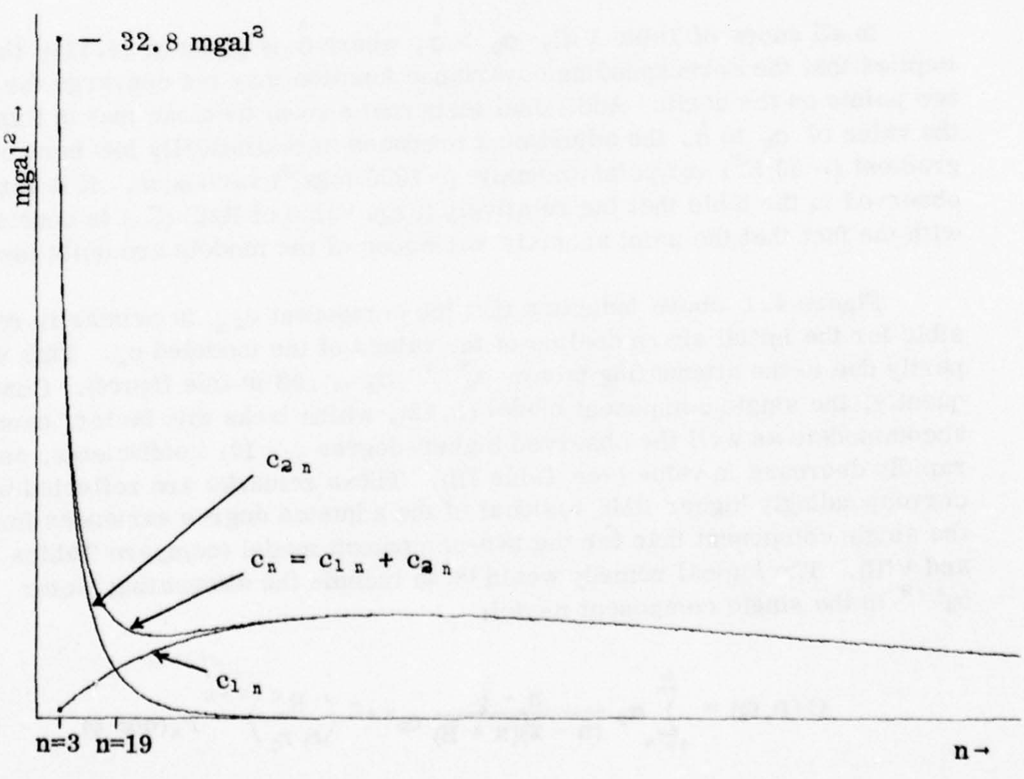


Figure 4.1: Model Degree Variances as a Function of n

Finally, it must be acknowledged and tests have shown that the computational effort in determining covariances with the model above (solution (i), Table VI) is approximately $3\frac{1}{2}$ times as great (due to the large value of A; see equ. (3.31)) as in the case of the model given by Tscherning and Rapp (1974) (equ. (3.12)).

IV.5 Data Set of Table III, Single Component Model

This latter model drew some criticism since it yields a relatively large horizontal gradient variance ($> 3000\ E^2$). For this model, there seems to exist a basic incompatibility between a point anomaly variance $C_0 \approx 1800\ \text{mgal}^2$ and a relatively low ($<600\ E^2$) gradient variance $G_{0\,H}$. Table VIII, which lists some solutions of the model (3.12) with the same observed data as previously (Tables III and V) verifies this conclusion. Note that in this model, c_n does not depend on σ_2, and therefore, the degree variances presumably refer to the Bjerhammar sphere (~ radius R_2) and are not modified as before. Again, none of the adjusted variances in Table VII includes the contribution of c_2. Also, while c_n is now independent of σ_2, the variances are not, and thereby, the capability of adjusting this parameter is retained.

In all cases of Table VIII, $\sigma_2 > \hat{\sigma}$, where $\hat{\sigma}$ is given by (4.11); this implies that the corresponding covariance function may not converge for every two points on the earth. Additional tests (not shown) disclose that in fixing the value of σ_2 to $\hat{\sigma}$, the adjustment produces unrealistically low horizontal gradient (~ $30\ E^2$) and point anomaly (~ $1000\ \text{mgal}^2$) variances. It is further observed in the table that the relatively large value of RMS ($\bar{C}_0$) is consistent with the fact that the point anomaly variances of the models are quite low.

Figure 4.1 above indicates that the component $c_{2\,n}$ is primarily responsible for the initial sharp decline of the values of the modeled c_n. This is partly due to the attenuating factor σ_2^{n+2} ($\sigma_2 \approx .93$ in this figure). Consequently, the single component model (3.12), which lacks this factor, cannot accommodate as well the observed higher-degree (> 10) coefficients, as these rapidly decrease in value (see Table III). These remarks are reflected by a correspondingly higher RMS residual of the adjusted degree variances for the single component than for the two-component model (compare Tables VI and VIII). The logical remedy would be to include the attenuating factor σ_2^{n+2} in the single component model:

$$C(P,Q) = \sum_{n=3}^{\infty} \alpha_2 \frac{n-1}{(n-2)(n+B)}\, \sigma_2^{n+2} \left(\frac{R_2^2}{r_P\, r_Q}\right)^{n+2} P_n(\cos\psi) \tag{4.13}$$

where $c_n = \alpha_2 \dfrac{n-1}{(n-2)\,(n+B)}\, \sigma_2^{n+2}$ now refers to the mean earth sphere.

Table VIII: Least Squares So;utions for the Model (3.12) with the Data of Tables III and V.

Sln.		stand. dev. if diff. from Table V (C_0) $(mgal)^2$	$(\bar{C}_{0\,1^\circ})$ $(mgal)^2$	$(\bar{C}_{0\,5^\circ})$ $(mgal)^2$	Model Parameters α_2 $(mgal)^2$	σ_2	Variances implied by the Model C_0 $(mgal)^2$	$G_{\alpha H}$ E^2	L_0 m^2	$\bar{C}_0(1^\circ)$ $(mgal)^2$	$\bar{C}_0(5^\circ)$ $(mgal)^2$	$RMS(\bar{C}_0)$ $(mgal)^2$	$RMS(c_n)$ $(mgal)^2$
(a)	B=18				290.7011	.9990624	1052.2	401.2	527.0	625.2	290.5	164.8	3.63
(b)	B=24	75			343.3408	.9988961	1098.5	338.9	490.6	649.4	286.6	147.5	3.90
(c)	B=24	50	3	3	408.8991	.9991130	1391.5	627.9	585.2	783.6	342.7	81.8	4.83
(d)	B=30	50	3	3	458.3376	.9990494	1435.4	608.9	542.0	792.1	331.3	71.7	4.90
(e)	B=30	50	1	1	465.0174	.9992834	1578.1	1093.8	550.8	815.7	337.7	66.8	5.01
(f)	B=27	50	1.7	1.7	440.0096	.9992071	1495.2	845.6	570.3	805.5	342.3	73.0	5.05
(g)	B=27	50	1.5	1.5	440.4454	.9992339	1510.8	907.3	571.0	807.6	342.8	72.6	5.06
(h)	B=24	50	1.7	1.7	414.9663	.9992325	1468.6	853.3	594.4	801.0	348.6	78.4	5.12

However, while the fit to the observed degree variances could improve, the resulting variances G_0 , C_0 , L_0 are then quite unnatural.

IV.6 Data Set of Table IV, Both Models

The remaining analysis is based on the observed degree variances in Table IV (from GEM 9 potential coefficients). However, the observed variances of Table V are retained. With this data set, and with each of the two horizontal gradient variances (200 E^2 , 3500 E^2), parameters were obtained for the two models (3.12) and (3.23). The identical procedure is followed in these determinations as with the previous data set. Therefore, only the final solutions of the four resulting models are presented in Table IX. Their adjustment was based on a set of observation equations which contained the corrected equation for the horizontal gradient variance $G_{0\,H}$, and from which the equation for the undulation variance L_0 had been deleted (see earlier remarks, sections III.4.1 and IV.2.3). Neither of these alterations has a very pronounced effect on the solution. The final weights for the variances C_0 and $\overline{C}_0$ were chosen primarily for reasons of consistency. The adjusted variances have been augmented by the value of the second-degree terms of their series expansions.

The first four solutions of Table IX are each designated by a number and a letter. The number refers to the number of components in the model, and the letter signifies whether the value of the horizontal gradient variance is high or low. For example, Model 2L corresponds to the two-component model (3.23) with an "observed" horizontal gradient variance of $G_{0\,H} = 200\ E^2$. Model TR in Table IX is the single component model obtained by Tscherning and Rapp (1974). It is included for comparison.

Table X below contains the adjusted degree variances of the models of Table IX. For the single component models, they refer to the respective spheres of radius R_2 (= $R_e\sqrt{\sigma_2}$), and for the two-component models, they refer to the mean earth sphere. The values of c_n' in Table IV are also repeated here.

Table IX: Final Least Squares Solutions of Models (3.23) and (3.12) for the Data of Tables IV (GEM 9) and V.

Model		init. value	stand. dev.		Model Parameters		adj. value	Variances implied by the Model plus the value of the c_2-term						
		G_{OH} ±sd. E^2	(C_0)	$(\bar{C}_0)$ $(mgal)^2$	α_1, α_2 $(mgal)^2$	σ_1, σ_2	c_3, c_{20} $(mgal)^2$	C_0 $(mgal)^2$	G_{OH} E^2	L_0 m^2	$\bar{C}_0(1^\circ)$ $(mgal)^2$	$\bar{C}_0(5^\circ)$ $(mgal)^2$	RMS$(\bar{C}_0)$ $(mgal)^2$	RMS(c_n) $(mgal)^2$
2L	A=100 B= 20	200± 10	25	3	18.3906 658.6132	.9943667 .9048949	35.1 4.5	1828.7	200.3	919.4	862.44	251.78	5.2	2.74
2H	A=140 B= 10	3500 ± 100	25	3	14.0908 160.6701	.9939083 .9997595	24.9 7.1	1870.2	3526.	779.4	856.95	275.48	12.2	4.68
1L	B= 30	200±1	25	3	491.1365	.9982959	29.8 10.4	1287.4	199.6	888.5	817.82	357.28	76.2	6.32
1H	B= 30	3500 ± 20	25	3	454.2862	.9996025	27.5 9.6	1802.6	3504.	852.1	820.86	339.39	63.9	5.83
TR*	B= 24	-	-	-	425.28	.999617	31.5 10.2	1795.0	3542.	926.1	848.00	367.32	77.1	6.28

* The parameters of Model TR had been determined by Tscherning and Rapp (1974).

Model 2L: two-component model (3.23) with an observed value of G_{OH} = 200 E^2

Model 2H: two-component model (3.23) with an observed value of G_{OH} = 3500 E^2

Model 1L: single component model (3.12) with an observed value of G_{OH} = 200 E^2

Model 1H: single component model (3.12) with an observed value of G_{OH} = 3500 E^2

Table X

Obs. c_n'		Adjusted Degree Variances (mgal^2)				
n	GEM 9	Model 2L	Model 2H	Model 1L	Model 1H	Model TR
3	33.66	35.1	24.9	29.8	27.5	31.5
4	19.63	23.1	17.5	21.7	20.0	22.8
5	20.87	18.1	14.6	18.7	17.3	19.6
6	19.05	15.1	13.0	17.1	15.8	17.7
7	19.45	12.9	11.9	15.9	14.7	16.5
8	11.73	11.2	11.0	15.1	13.9	15.5
9	11.50	9.9	10.3	14.4	13.3	14.7
10	10.07	8.9	9.8	13.8	12.8	14.1
11	6.77	8.0	9.3	13.3	12.3	13.5
12	3.67	7.3	8.9	12.9	11.9	13.0
13	6.59	6.7	8.6	12.5	11.5	12.5
14	4.04	6.2	8.3	12.1	11.2	12.1
15	3.30	5.7	8.0	11.8	10.9	11.7
16	2.34	5.4	7.8	11.4	10.6	11.4
17	2.05	5.1	7.6	11.1	10.3	11.1
18	3.32	4.9	7.4	10.9	10.1	10.8
19	2.99	4.7	7.2	10.6	9.8	10.5
20	2.30	4.5	7.1	10.4	9.6	10.2
*	R_e	6371.km	6371.km	6365.57km	6369.73km	6369.78km

* the radius to which the degree variances refer

Moritz (1976) discusses the three "essential parameters" which characterize the covariance function for gravity anomalies locally. Two of these, the variance and the curvature at $\psi = 0$ (related also to the horizontal gradient variance, see p. 18) have already been determined for all models above. The third parameter, being the correlation length ξ, cannot be included conveniently into the adjustment. The solution to the equation $C(\xi) = \frac{1}{2} C_0$ (see p. 18) is found by fifth-order polynomial inverse interpolation. The results are tabulated below for the five models of Table IX.

Table XI: Correlation Lengths ξ

Model	Correlation Length (km)
2L	46.087
2H	43.693
1L	86.856
1H	38.866
TR	42.284

The covariance functions C (P, Q), L(P, Q), G(P, Q) for the first four models are evaluated at selected values of ψ between 0° and 180° using the closed expressions (3.48), (3.53), (3.54), (3.44) and their auxiliary formulas. Tables XII, XIII, XIV display the various results.

The mean anomaly covariances, on the other hand, are computed from the approximate formula (3.60). Tables XV and XVI list 1° and 5° mean anomaly covariances, respectively, for several values of the argument ψ.

The graphs of the covariance functions (Figures 4.2, 4.3, 4.4, 4.5) are designed to depict the contrasts of the functions implied by the models near the origin ($\psi = 0$) where they differ the most. Whence, the abscissa is scaled logarithmically. Also, all functions represent covariances on the mean earth sphere.

For applications in a limited, local area, it is often practical to determine the anomalous quantities of the gravity field with respect to a reference field of higher degree and order than implied by a rotational ellipsoid. The local covariance function then does not include the low-degree (long-wavelength) information of the field. In regard to the series expansions, the first few terms up to degree j are deleted yielding a j^{th}-order covariance function, for example

$$K_j(P,Q) = \sum_{n=j+1}^{\infty} k_n \left(\frac{R^2}{r_P r_Q}\right)^{n+1} P_n(\cos\psi)$$

$$= K(P,Q) - \sum_{n=2}^{j} k_n \left(\frac{R^2}{r_P\, r_Q}\right)^{n+1} P_n(\cos\psi) \tag{4.14}$$

where the k_n, $n = 2, \ldots, j$ are computed from the model.

For the first four models of Table IX, covariances of this type were examined with $j = 20$. As expected, the removal of the low-degree terms has no significant influence on the gradient covariances. Also, the contrasts in the modeled higher-order anomaly covariances does not diminish radically as their variances all decrease by approximately 200 to 250 $mgal^2$. The undulation covariances of this 20^{th}-order field, on the other hand, are practically indistinguishable (to 10 m^2) with respect to the four models; the variances decrease to $\sim 315\ m^2$. Therefore, the choice of the models above is not critical when higher-order undulation covariances are required.

TABLE XII: POINT ANOMALY COVARIANCES ON THE SPHERE OF RADIUS RE=6371. KM, IN MGAL**2

PSI(DEG)	MODEL 1L	MODEL 1H	MODEL 2L	MODEL 2H
0.0	1287.419	1802.639	1828.651	1870.191
0.25	1001.888	1025.346	1311.206	1215.059
0.50	788.926	771.887	766.122	781.443
0.75	656.834	632.005	498.853	556.970
1.00	565.272	539.162	368.008	436.459
1.25	497.193	471.684	298.119	364.348
1.50	444.168	419.858	257.491	316.738
1.75	401.479	378.528	232.058	282.754
2.00	366.249	344.651	215.052	257.004
3.00	270.135	253.130	180.792	193.509
4.00	212.306	198.590	163.055	157.236
5.00	173.307	161.990	149.071	132.388
10.00	81.073	75.860	94.142	68.475
15.00	43.241	40.636	56.765	38.766
20.00	21.782	20.651	31.325	20.797
25.00	7.900	7.702	13.528	8.737
30.00	-1.531	-1.117	0.947	0.331
35.00	-7.922	-7.118	-7.800	-5.504
40.00	-12.035	-11.006	-13.565	-9.374
45.00	-14.348	-13.224	-16.932	-11.674
50.00	-15.208	-14.091	-18.347	-12.695
55.00	-14.898	-13.868	-18.183	-12.677
60.00	-13.665	-12.784	-16.770	-11.830
65.00	-11.739	-11.049	-14.417	-10.352
70.00	-9.334	-8.862	-11.411	-8.426
75.00	-6.652	-6.406	-8.022	-6.225
80.00	-3.876	-3.854	-4.497	-3.911
85.00	-1.175	-1.360	-1.061	-1.626
90.00	1.305	0.941	2.092	0.500
95.00	3.443	2.936	4.799	2.362
100.00	5.144	4.536	6.934	3.875
105.00	6.341	5.680	8.408	4.980
110.00	6.997	6.331	9.171	5.640
115.00	7.104	6.481	9.213	5.844
120.00	6.680	6.146	8.559	5.607
125.00	5.771	5.365	7.268	4.961
130.00	4.443	4.199	5.428	3.960
135.00	2.781	2.725	3.154	2.674
140.00	0.884	1.033	0.575	1.183
145.00	-1.140	-0.780	-2.165	-0.423
150.00	-3.180	-2.613	-4.920	-2.054
155.00	-5.127	-4.365	-7.544	-3.618
160.00	-6.879	-5.944	-9.901	-5.030
165.00	-8.345	-7.266	-11.871	-6.214
170.00	-9.448	-8.262	-13.353	-7.107
175.00	-10.134	-8.881	-14.273	-7.663
180.00	-10.366	-9.091	-14.585	-7.851

TABLE XIII: UNDULATION COVARIANCES ON THE SPHERE OF RADIUS RE=6371. KM, IN M**2

PSI(DEG)	MODEL 1L	MODEL 1H	MODEL 2L	MODEL 2H
0.0	888.462	852.124	919.437	779.382
0.25	888.178	851.806	919.056	779.029
0.50	887.431	851.038	918.140	778.176
0.75	886.328	849.940	916.957	777.038
1.00	884.931	848.571	915.620	775.708
1.25	883.281	846.969	914.167	774.229
1.50	881.405	845.159	912.610	772.618
1.75	879.326	843.163	910.951	770.887
2.00	877.061	840.995	909.190	769.041
3.00	866.403	830.846	901.073	760.603
4.00	853.605	818.713	891.183	750.622
5.00	839.049	804.945	879.513	739.257
10.00	747.269	718.299	797.370	665.880
15.00	635.187	612.433	685.610	573.172
20.00	513.082	496.875	556.850	469.718
25.00	388.488	378.654	421.353	362.216
30.00	267.411	263.406	287.374	256.125
35.00	154.649	155.662	161.476	155.883
40.00	53.909	58.946	48.692	64.985
45.00	-32.141	-24.177	-47.360	-13.973
50.00	-101.819	-92.053	-124.391	-79.256
55.00	-154.366	-143.888	-181.338	-129.936
60.00	-189.857	-179.665	-218.240	-165.824
65.00	-209.104	-200.059	-236.098	-187.388
70.00	-213.533	-206.322	-236.709	-195.653
75.00	-205.055	-200.168	-222.494	-192.093
80.00	-185.926	-183.644	-196.309	-178.510
85.00	-158.608	-159.002	-161.267	-156.924
90.00	-125.633	-128.576	-120.551	-129.449
95.00	-89.479	-94.661	-77.255	-98.190
100.00	-52.456	-59.416	-34.237	-65.145
105.00	-16.608	-24.766	6.009	-32.116
110.00	16.356	7.659	41.445	-0.652
115.00	45.115	36.578	70.553	28.009
120.00	68.767	61.087	92.373	52.958
125.00	86.832	80.662	106.563	73.633
130.00	99.235	95.150	113.074	89.804
135.00	106.266	104.733	112.696	101.548
140.00	108.532	109.882	106.386	109.205
145.00	106.883	111.295	95.470	113.328
150.00	102.337	109.832	81.482	114.622
155.00	95.998	106.434	66.049	113.875
160.00	88.972	102.055	50.781	111.895
165.00	82.289	97.587	37.163	109.442
170.00	76.831	93.799	26.459	107.176
175.00	73.273	91.278	19.634	105.603
180.00	72.039	90.396	17.290	105.043

TABLE XIV: VERTICAL GRADIENT COVARIANCES ON THE SPHERE OF RADIUS RE=6371. KM, IN E**2

PSI(DEG)	MODEL 1L	MODEL 1H	MODEL 2L	MODEL 2H
0.0	399.23237	7007.93183	400.50616	7052.16421
0.25	15.29288	0.66253	67.79675	45.35111
0.50	0.91263	-1.18474	-12.27541	-7.29198
0.75	-0.23575	-0.82614	-8.83743	-5.81297
1.00	-0.34166	-0.56790	-4.46769	-2.98748
1.25	-0.30362	-0.40621	-2.28538	-1.54193
1.50	-0.24996	-0.30142	-1.22568	-0.84185
1.75	-0.20298	-0.23044	-0.68590	-0.48807
2.00	-0.16529	-0.18048	-0.39557	-0.29891
3.00	-0.07914	-0.08025	-0.04077	-0.06439
4.00	-0.04294	-0.04230	0.01138	-0.02211
5.00	-0.02548	-0.02472	0.01775	-0.01005
10.00	-0.00393	-0.00370	0.00474	-0.00106
15.00	-0.00133	-0.00124	0.00087	-0.00045
20.00	-0.00089	-0.00081	-0.00024	-0.00047
25.00	-0.00085	-0.00078	-0.00072	-0.00059
30.00	-0.00090	-0.00082	-0.00098	-0.00069
35.00	-0.00094	-0.00086	-0.00112	-0.00076
40.00	-0.00095	-0.00087	-0.00118	-0.00079
45.00	-0.00092	-0.00085	-0.00116	-0.00077
50.00	-0.00085	-0.00079	-0.00109	-0.00073
55.00	-0.00075	-0.00070	-0.00097	-0.00065
60.00	-0.00063	-0.00059	-0.00082	-0.00056
65.00	-0.00049	-0.00046	-0.00064	-0.00044
70.00	-0.00035	-0.00033	-0.00045	-0.00032
75.00	-0.00020	-0.00019	-0.00025	-0.00019
80.00	-0.00005	-0.00006	-0.00007	-0.00007
85.00	0.00008	0.00006	0.00011	0.00004
90.00	0.00019	0.00017	0.00026	0.00014
95.00	0.00028	0.00025	0.00038	0.00022
100.00	0.00035	0.00032	0.00046	0.00028
105.00	0.00039	0.00035	0.00051	0.00032
110.00	0.00040	0.00037	0.00052	0.00033
115.00	0.00038	0.00035	0.00050	0.00032
120.00	0.00034	0.00031	0.00044	0.00029
125.00	0.00027	0.00025	0.00035	0.00024
130.00	0.00019	0.00018	0.00024	0.00017
135.00	0.00009	0.00009	0.00010	0.00009
140.00	-0.00002	-0.00001	-0.00005	-0.00000
145.00	-0.00014	-0.00012	-0.00020	-0.00010
150.00	-0.00025	-0.00022	-0.00035	-0.00019
155.00	-0.00036	-0.00032	-0.00049	-0.00028
160.00	-0.00045	-0.00040	-0.00062	-0.00036
165.00	-0.00053	-0.00048	-0.00073	-0.00042
170.00	-0.00059	-0.00053	-0.00081	-0.00047
175.00	-0.00063	-0.00056	-0.00085	-0.00050
180.00	-0.00064	-0.00057	-0.00087	-0.00051

TABLE XV : MEAN (1 DEG) ANOMALY COVARIANCES ON THE SPHERE OF RADIUS RE=6371. KM, IN MGAL**2

PSI(DEG)	MODEL 1L	MODEL 1H	MODEL 2L	MODEL 2H
0.0	817.818	820.863	862.442	856.950
0.25	792.764	791.735	811.918	814.074
0.50	729.049	719.899	687.597	708.122
0.75	647.916	630.986	540.486	580.679
1.00	567.895	545.457	412.345	465.908
1.25	500.625	476.442	324.375	382.224
1.50	447.109	423.394	271.632	326.617
1.75	403.878	381.210	239.830	288.394
2.00	368.207	346.755	219.476	260.418
3.00	271.075	254.070	181.301	194.258
4.00	212.822	199.093	163.019	157.519
5.00	173.619	162.289	148.964	132.535
10.00	81.128	75.912	94.128	68.502
15.00	43.261	40.654	56.771	38.778
20.00	21.792	20.660	31.333	20.804
25.00	7.907	7.709	13.586	8.743
30.00	-1.525	-1.112	0.954	0.336
35.00	-7.917	-7.114	-7.794	-5.499
40.00	-12.031	-11.002	-13.559	-9.370
45.00	-14.344	-13.220	-16.927	-11.671
50.00	-15.205	-14.087	-18.342	-12.692
55.00	-14.895	-13.865	-18.178	-12.674
60.00	-13.663	-12.782	-16.767	-11.828
65.00	-11.737	-11.048	-14.415	-10.350
70.00	-9.333	-8.861	-11.410	-8.425
75.00	-6.651	-6.406	-8.021	-6.225
80.00	-3.876	-3.855	-4.497	-3.911
85.00	-1.176	-1.361	-1.062	-1.627
90.00	1.304	0.940	2.090	0.499
95.00	3.442	2.934	4.797	2.361
100.00	5.142	4.535	6.931	3.874
105.00	6.339	5.678	8.405	4.978
110.00	6.995	6.330	9.169	5.638
115.00	7.102	6.480	9.211	5.843
120.00	6.679	6.145	8.557	5.606
125.00	5.779	5.364	7.266	4.960
130.00	4.442	4.199	5.422	3.960
135.00	2.781	2.725	3.154	2.674
140.00	0.885	1.033	0.575	1.183
145.00	-1.139	-0.780	-2.164	-0.423
150.00	-3.179	-2.612	-4.918	-2.053
155.00	-5.126	-4.364	-7.541	-3.617
160.00	-6.877	-5.942	-9.898	-5.028
165.00	-8.342	-7.263	-11.867	-6.212
170.00	-9.446	-8.259	-13.349	-7.105
175.00	-10.131	-8.878	-14.269	-7.660
180.00	-10.363	-9.088	-14.581	-7.849

TABLE XVI: MEAN (5 DEG) ANOMALY COVARIANCES ON THE SPHERE OF RADIUS RE=6371. KM, IN MGAL**2

PSI(DEG)	MODEL 1L	MODEL 1H	MODEL 2L	MODEL 2H
0.0	357.277	339.387	251.781	275.477
0.25	356.427	338.553	251.163	274.727
0.50	353.904	336.084	249.344	272.510
0.75	349.796	332.066	246.423	268.929
1.00	344.234	326.634	242.568	264.137
1.25	337.381	319.956	237.949	258.321
1.50	329.421	312.215	232.762	251.678
1.75	320.538	303.597	227.179	244.398
2.00	310.905	294.274	221.344	236.646
3.00	267.771	252.724	196.990	203.137
4.00	222.740	209.515	172.450	168.964
5.00	182.386	170.964	150.815	136.974
10.00	82.533	77.230	93.734	69.189
15.00	43.734	41.094	56.900	39.059
20.00	22.037	20.887	31.517	20.974
25.00	8.077	7.865	13.715	8.873
30.00	-1.385	-0.983	1.129	0.449
35.00	-7.794	-7.000	-7.635	-5.396
40.00	-11.919	-10.899	-13.412	-9.275
45.00	-14.244	-13.127	-16.794	-11.584
50.00	-15.117	-14.006	-18.225	-12.615
55.00	-14.821	-13.796	-18.079	-12.609
60.00	-13.604	-12.726	-16.683	-11.775
65.00	-11.694	-11.007	-14.357	-10.311
70.00	-9.306	-8.835	-11.378	-8.399
75.00	-6.639	-6.394	-8.005	-6.212
80.00	-3.879	-3.856	-4.500	-3.910
85.00	-1.191	-1.374	-1.082	-1.637
90.00	1.278	0.916	2.056	0.479
95.00	3.407	2.903	4.752	2.383
100.00	5.102	4.498	6.879	3.841
105.00	6.296	5.639	8.349	4.943
110.00	6.952	6.290	9.113	5.602
115.00	7.062	6.443	9.159	5.899
120.00	6.644	6.113	8.512	5.576
125.00	5.743	5.339	7.232	4.987
130.00	4.425	4.183	5.406	3.944
135.00	2.775	2.719	3.147	2.668
140.00	0.891	1.038	0.584	1.187
145.00	-1.121	-0.764	-2.139	-0.409
150.00	-3.148	-2.585	-4.877	-2.029
155.00	-5.084	-4.326	-7.485	-3.584
160.00	-6.825	-5.896	-9.828	-4.987
165.00	-8.282	-7.209	-11.787	-6.164
170.00	-9.379	-8.200	-13.260	-7.052
175.00	-10.061	-8.815	-14.175	-7.604
180.00	-10.292	-9.024	-14.485	-7.791

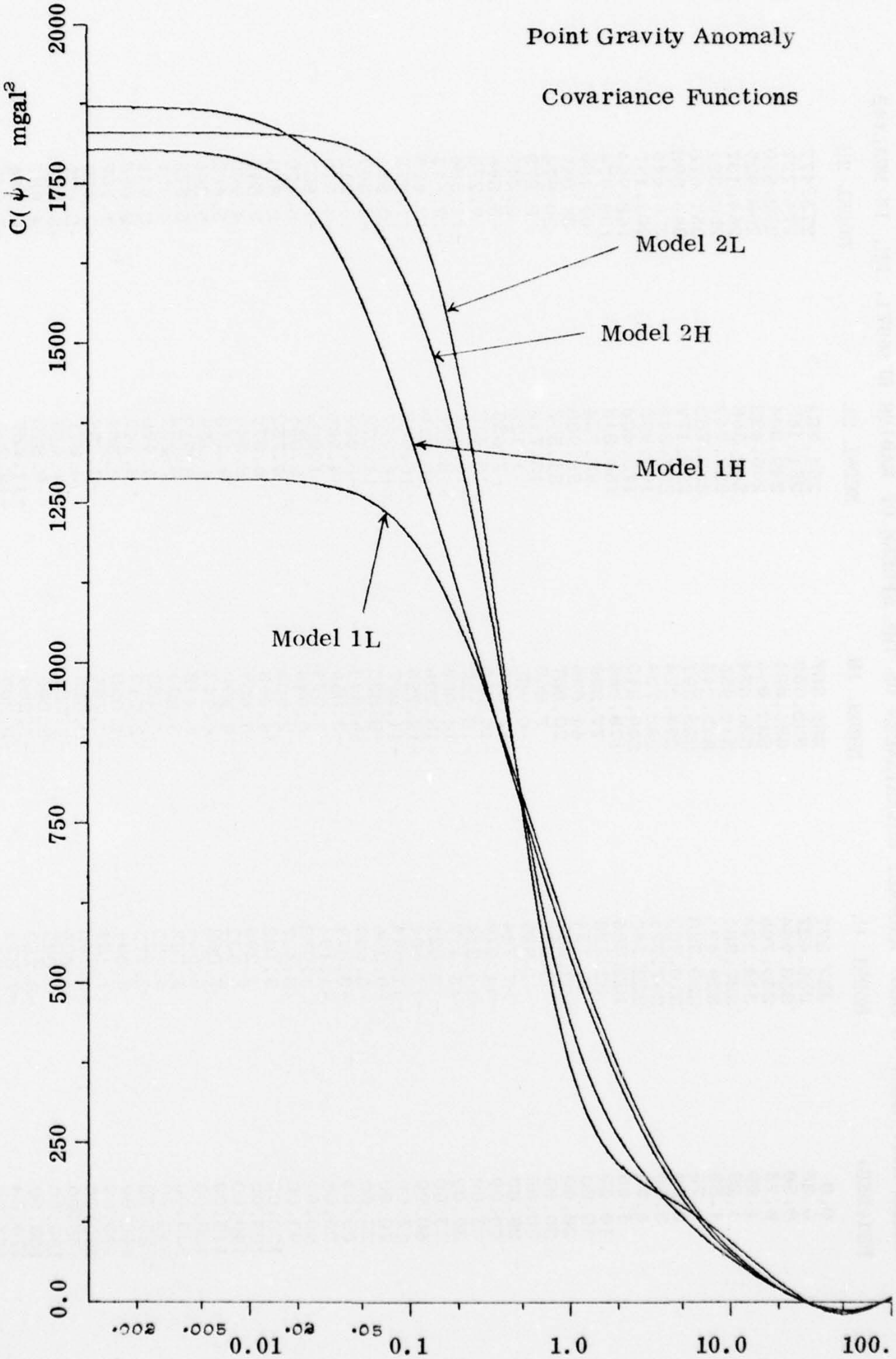
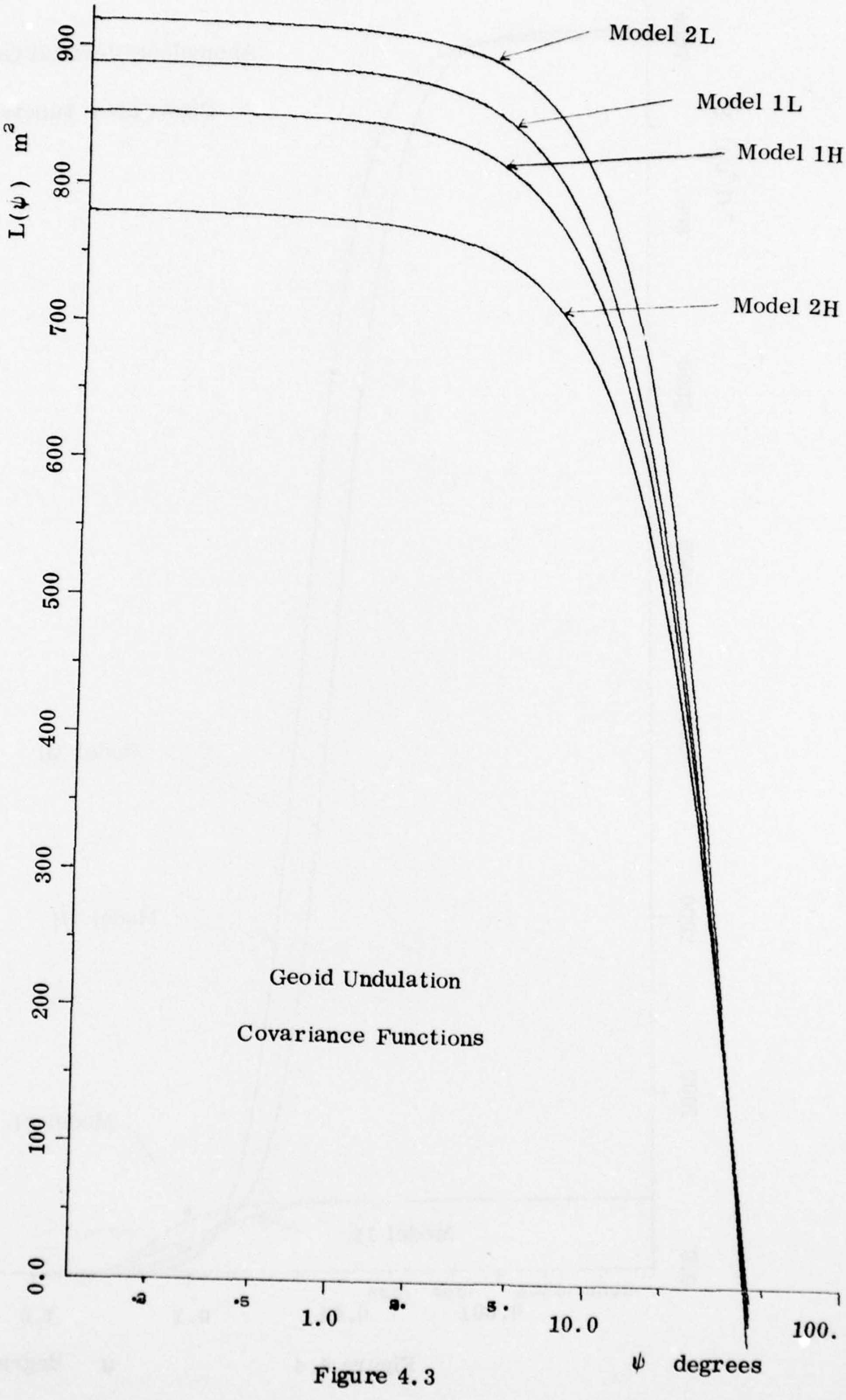
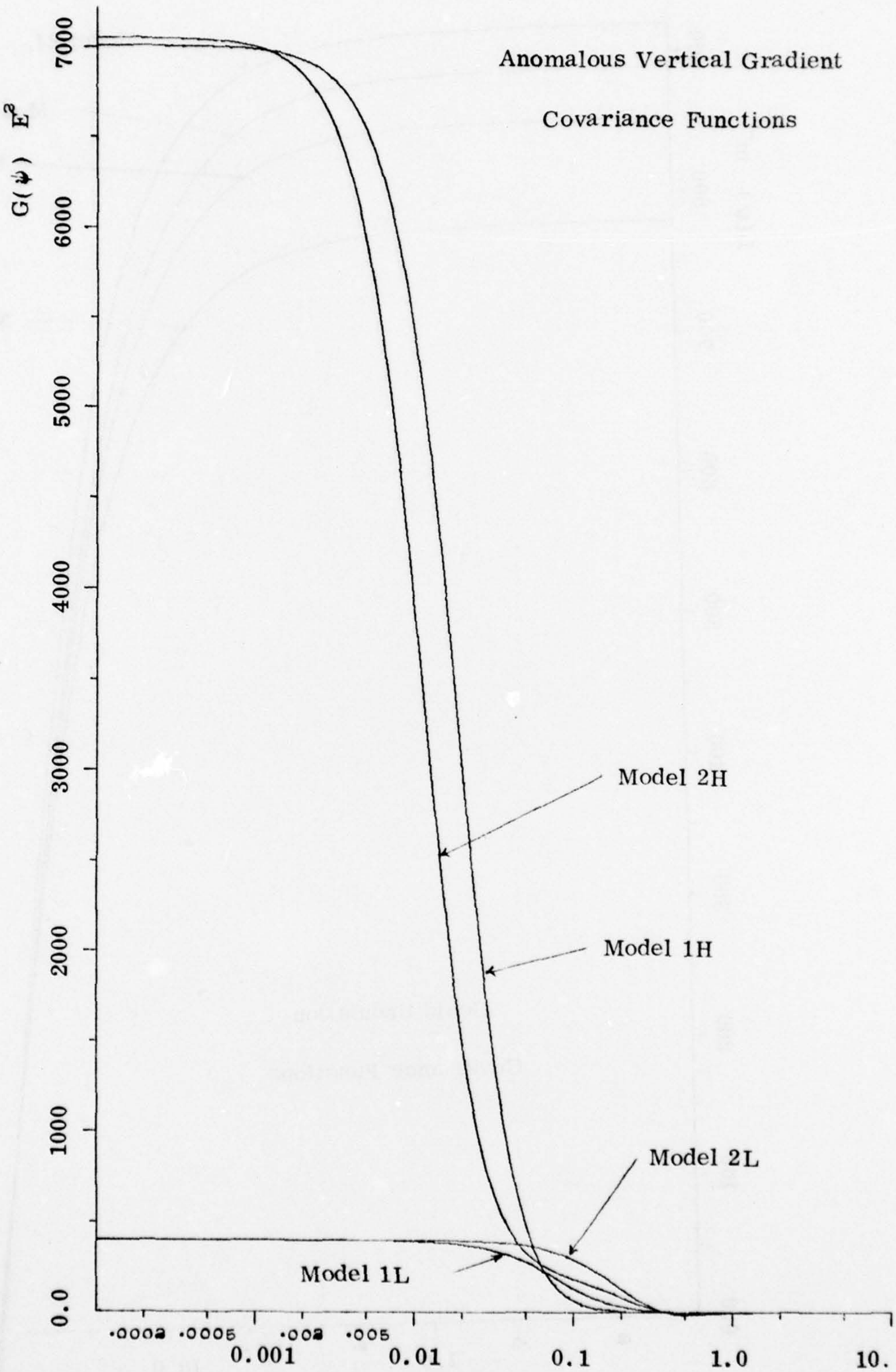

Point Gravity Anomaly
Covariance Functions
$C(\psi)$ mgal^2
2000
1750
1500
1250
1000
750
500
250
0.0
Model 2L
Model 2H
Model 1H
Model 1L
.002
.005
0.01
.02
.05
0.1
1.0
10.0
100.
ψ degrees

Figure 4.2



Model 2L
Model 1L
Model 1H
Model 2H
L(ψ) m²
900
800
700
600
500
400
300
200
100
0.0
Geoid Undulation
Covariance Functions
.5
1.0
5.
10.0
100.
ψ degrees

Figure 4.3



Anomalous Vertical Gradient
Covariance Functions
G(ψ) E²
7000
6000
5000
4000
3000
2000
1000
0.0
Model 2H
Model 1H
Model 2L
Model 1L
.0002
.0005
.002
.005
0.001
0.01
0.1
1.0
10.
ψ degrees

Figure 4.4

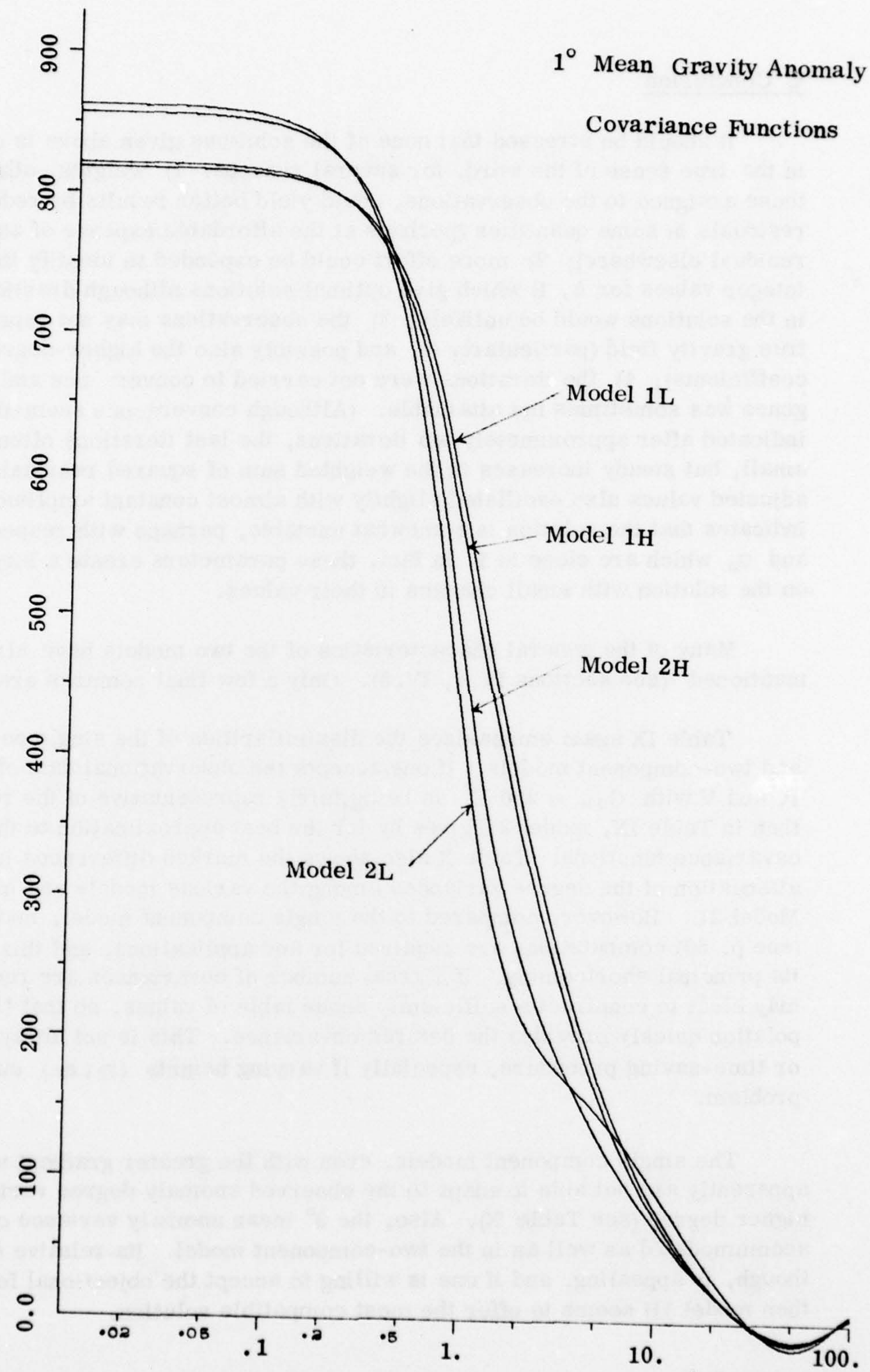

1° Mean Gravity Anomaly
Covariance Functions
$\bar{C}(\psi)$ mgal^2
900
800
700
600
500
400
300
200
100
0.0
Model 1L
Model 1H
Model 2H
Model 2L
.02
.05
.1
.2
.5
1.
10.
100.
ψ degrees

Figure 4.5

V. Conclusion

It should be stressed that none of the solutions given above is optimal, in the true sense of the word, for several reasons: 1) weights, other than those assigned to the observations, could yield better results by reducing the residuals in some quantities (perhaps at the affordable expense of an increased residual elsewhere); 2) more effort could be expended to identify the exact integer values for A, B which give optimal solutions although drastic changes in the solutions would be unlikely; 3) the observations may not represent the true gravity field (particularly G_0 and possibly also the higher-degree anomaly coefficients); 4) the iterations were not carried to convergence and convergence was sometimes not attainable. (Although convergence seemed to be indicated after approximately ten iterations, the last iterations often exhibited small, but steady increases in the weighted sum of squared residuals.) Some adjusted values also oscillated slightly with almost constant amplitude. This indicates that the solution is somewhat unstable, perhaps with respect to σ_1 and σ_2 which are close to 1. In fact, these parameters create a large impact on the solution with small changes in their values.

Many of the general characteristics of the two models have already been mentioned (see sections IV.4, IV.5). Only a few final remarks are noteworthy.

Table IX again emphasizes the dissimilarities of the single component and two-component models. If one accepts the observational data of Tables IV and V with $G_{0H} \approx 200\ E^2$ as being fairly representative of the real world, then in Table IX, model 2L gives by far the best approximation to the global cavariance functions. Table X also shows the marked differences in the attenuation of the degree variances among the various models and in favor of Model 2L. However, compared to the single component model, many more (see p. 50) computations are required for any applications, and this constitutes its principal shortcoming. If a great number of covariances are required, one may elect to construct a sufficiently dense table of values, so that linear interpolation quickly provides the desired covariance. This is not always a feasible or time-saving procedure, especially if varying heights (r_P, r_Q) enter the problem.

The single component models, even with the greater gradient variance, apparently are not able to adapt to the observed anomaly degree variances of higher degree (see Table X). Also, the 5° mean anomaly variance cannot be accommodated as well as in the two-component model. Its relative simplicity, though, is appealing, and if one is willing to accept the objectional features, then model 1H seems to offer the most compatible solution.

Certainly, if new observational data proves to be more characteristic of the actual attributes of the gravity field, then the single component model may attain better suitability. Even so, the two-component model with twice as many parameters will have the greater capability of adjusting to any new, improved data.

Appendix: Derivation of equation (2.76): $G_{OH} = \frac{1}{2} G_O$

Let x, y, z be a local Cartesian coordinate system with the z-axis coinciding with the local vertical. Then $\frac{\partial \Delta g}{\partial x}$, $\frac{\partial \Delta g}{\partial y}$ are the horizontal gradients and $\frac{\partial \Delta g}{\partial z}$ is the vertical gradient. If T is the disturbing potential, then (roughly consistent with a planar approximation)

$$\Delta g = -\frac{\partial T}{\partial z}, \quad \text{and} \quad \frac{\partial \Delta g}{\partial z} = -\frac{\partial^2 T}{\partial z^2} \tag{A.1}$$

Moritz (1976) shows that $\frac{\partial^2 K}{\partial x_P \partial x_Q}$, $\frac{\partial^2 K}{\partial y_P \partial y_Q}$, $\frac{\partial^2 K}{\partial z_P \partial z_Q}$, being the covariance functions of $\frac{\partial T}{\partial x}$, $\frac{\partial T}{\partial y}$, and $\frac{\partial T}{\partial z}$, respectively, where K (P, Q) is the covariance function of T, satisfy the following simple relationship

$$\frac{\partial^2 K}{\partial z_P \partial z_Q} = \frac{\partial^2 K}{\partial x_P \partial x_Q} + \frac{\partial^2 K}{\partial y_P \partial y_Q} \qquad \text{(planar approximation)} \tag{A.2}$$

Similarly if F_{zz}, F_{zx}, F_{zy} are the covariance functions of $\frac{\partial}{\partial z}\left(\frac{\partial T}{\partial z}\right)$, $\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial z}\right)$, $\frac{\partial}{\partial y}\left(\frac{\partial T}{\partial z}\right)$, then

$$F_{zz} = F_{zx} + F_{zy} \tag{A.3}$$

Now, to the same approximation, it can be shown that the following statements are true (Moritz 1976)

$$\frac{\partial^2 K}{\partial z_P \partial z_Q} = -K''(s) - \frac{1}{s} K'(s) \tag{A.4}$$

$$\frac{\partial^2 K}{\partial x_P \partial x_Q} = -K''(s) \tag{A.5}$$

$$\frac{\partial^2 K}{\partial y_P \partial y_Q} = -\frac{1}{s} K'(s) \tag{A.6}$$

where s is linear distance and the primes denote differentiation with respect to s. At s = 0, the three left-hand expressions above are the gradient variances. We have

$$K'(s) = -sK'(s) - s\frac{\partial^2 K}{\partial z_P \partial z_Q} \tag{A.7}$$

If it is assumed that the variances $\left.\frac{\partial^2 K}{\partial x_P \partial x_Q}\right|_{s=0}$ and $\left.\frac{\partial^2 K}{\partial y_P \partial y_Q}\right|_{s=0}$ are finite then

$$K'(0) = 0 \tag{A.8}$$

Now,

$$K''(0) = \lim_{s \to 0} \frac{K'(s) - K'(0)}{s} = \lim_{s \to 0} \frac{1}{s} K'(s) < \infty \tag{A.9}$$

by the same assumption. Hence

$$\left.\frac{\partial^2 K}{\partial x_P \partial x_Q}\right|_{s=0} = \left.\frac{\partial^2 K}{\partial y_P \partial y_Q}\right|_{s=0} \tag{A.10}$$

Then also $F_{zx}\big|_{s=0} = F_{zy}\big|_{s=0}$ and equation (A.3) states that the horizontal gradient variance is one half of the vertical gradient variance (in a planar approximation).

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